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Exploring The Spectrum: A Comparative Analysis Of Eigenvalues In Diverse Graph Structures Within Graph Theory

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ABSTRACT

This comparative analysis investigates eigenvalues of graph theory based on scale-free, random, and regular graphs in an effort to comprehend their structural characteristics. Eigenvalues give useful information on network activity and hence affect applications such as resilience, connectivity, and randomness. Regular graphs have standard eigenvalue characteristics because of the homogenization of node degrees, and random graphs have unordered spectral distributions consistent with their randomness. Scale-free graphs characterized by power-law degree distributions exhibit how high-degree nodes determine eigenvalue landscapes in a bid to understand network stability.

By studying actual networks—social, biological, and technical networks—this paper closes the gap between theory and practice. The results affect network design, optimization, and anomaly detection and provide insights to researchers, engineers, and practitioners. This research also advances our knowledge of spectral properties in general graph structures to lay the groundwork for future research and innovative application in complex system analysis.

Keywords: *Spectral Signatures, Graph Characterization, Eigenvalues, Graph Theory, Topological Characteristics, Network Dynamics, Practical Applications.*

1. INTRODUCTION

Relationships between structures and eigenvalues in spectral theory graphs

The investigation of the connections that exist between eigenvalues and graph topologies is the fundamental work that underpins spectral graph theory. In order to make use of spectral graph theory, it is necessary to have previous understanding of both graph theory and linear algebra principles. Determinants, eigenvalues, and eigenvectors are some of the concepts that fall under this category. Other concepts include the Courant-Fischer and Perron-Frobenius theorems.

The matrix of adjacency

First, the adjacency matrix should be produced, and then the vertices of graph G should be used to index the columns and rows of the graph. The result of this will be matrix A, which is also referred to as the adjacency matrix.

$$A_{i,j} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j; \\ 0 & \text{if } i \text{ is not adjacent to } j. \end{cases}$$

As an example, the adjacency matrix seen in Figure 1 comprises the graph

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The adjacency matrix is not really unique since it is possible to alter the names of the vertices of the graph, which would make it possible to alter both the rows and the columns at the same time. One example of an adjacency matrix that might have been derived from the graph is the matrix that is shown below:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

It is crucial to acknowledge that regardless of the relabeling, the eigenvalues of the two matrices will remain unchanged. This is a matter that requires attention. In addition, when we associate eigenfunctions, which can be seen as functions on the vertices, with the vertices, we can conclude that the eigenfunctions are unaffected by the choice of labels, without requiring any additional clarification. (Singh, H., & Sharma, R,2012)

Spectra of graphs produced by different graphs and their uses in a variety of fields

The field of research known as spectral graph theory investigates the ways in which the structural characteristics of a network are linked to the spectra of certain matrices that are connected to the network. In the collection of published literature, the spectrum may be retrieved from a wide variety of matrices that are related to graphs. The adjacency matrix, the Laplacian matrix, and the signless Laplacian matrix are all examples of matrices that are often used in research. Various spectrum features of these matrices have been investigated by a large number of independent researchers. It is possible to find applications for the study of spectra in a wide range of subjects, including as computer science, information and communication technologies, biology, geography, economics, and social sciences (for example, see the references made within this article). The study of graph spectrum analysis encompasses a wide range of topics. In the following paragraphs, you will find a description of some of the specific topics that this website attempts to investigate in relation to this discipline.

Every single graph that is taken into consideration in this article is presumed to be straightforward and connected, unless it is specifically stated differently. Imagine a graph G that has n edges with the values 1, 2,... Imagine this graph. E(G) is a more typical method to define the set of edges of G, while V(G) is a shorthand for the set of vertices of G. V(G) denotes the set of vertices of G. In the graph G, the symbol i ∼ j signifies that there exists a connection or edge between the vertices i and j when it is present. Specifically, the adjacency matrix of G is represented by the matrix A(G), which is a matrix of n × n elements. (Cvetković, D., & Simić, S. K.,2010)

One may ascertain if aij has a value of 1 or 0 by determining whether or not i is a member of j. This is a fact that is widely distributed. The topics of graph adjacency matrices and Laplacian matrices have been the subject of a great number of books and papers respectively. It is highly recommended that you study two overview articles written by Merris and Mohar, as well as a classic book written by Cvetković, Doob, and Sachs, in order to get further historical knowledge on these two matrices. (Mukherjee, C,2014)

Contrasting the adjacency matrix and Laplacian spectrums

Employ the adjacency matrix for properties that are significant for enumerating walks and adjacency; the combinatorial Laplacian should be utilized for problems that may involve spanning trees or the incidence of vertices and edges; the normalized Laplacian should be utilized for problems that may involve random walks; and finally, the adjacency matrix should be utilized for problems that may involve spanning trees. In the most recent years, the normalized Laplacian has gained a greater amount of popularity. The majority of this is due to the fact that random walks have the potential to be used in the process of efficiently searching through enormous datasets. (Merris, R,1994).

If we were to demonstrate that the eigenvalues of equations (1.1), (1.2), and (1.3) are considerably different from one another, then it would be acceptable to believe that this is the case. A broader point of view is that discoveries that are applicable to one spectrum may not be applicable to another spectrum, and the selection of the spectrum will have a significant influence on the final result. major exceptions to this rule include the fact that, in the case of regular graphs, the spectra of all three graphs are, in fact, scaled and shifted replicas of each other. This is one of the few major exceptions to this rule. This is due to the fact that regular graphs $L = k\mathcal{L} = k\left(I - \frac{1}{k}A\right).$

Due to the fact that there are three different matrices to choose from, it may be difficult to generalize from regular graphs to generic graphs. This is due to the fact that, in contrast to generic graphs, normal graphs include the inclusion of more specific information.

There are not many individuals who are aware of the fact that the eigenvalue of the adjacency matrix is zero for a network that does not include any isolated vertices. This is the same as the eigenvalue of the normalized Laplacian, which has the value of one. In the adjacency matrix, the number of negative eigenvalues is equal to the number of positive eigenvalues,

which are eigenvalues that are less than 1. This is true regardless of the number of normalized Laplacian eigenvalues that are more than 1. After this, we shall proceed to a straightforward illustration of how Sylvester's Law of Inertia may be used in practice.
Despite the fact that the three spectra do not scale clearly for generic graphs (and as a result, their "shape" could be different), we are able to demonstrate that the spectra of the combinatorial Laplacian and the normalized Laplacian are comparable when the graph is approximately regular.

When it comes to the vertex sets of graphs G and G c, which is the complement of graph G, there is no distinction between the two. The fact that two vertices in G c are only considered to be close to one another if they are not next to one another in G is another well-known fact. G1 ∧ G2 is the notation that is used to represent the graph that is created when two graphs, G1 and G2, are combined. Graph vertex names are represented by the set V(G1) ∧ V(G2), whereas the set of graph edges is represented by the set E(G1) ∧ E(G2).

The join of graphs G1 and G2, also known as G1 ∨ G2, is the graph that is produced by connecting all of the possible edges from the vertices of graph G1 to those in vector G2. In spite of this, the graph that was constructed is referred to as G1 ∨ G2. It is possible that the join operation is also referred to as complete product in certain quarters. With the assistance of the graph products, it is possible to easily build a wide variety of significant graphs of varying purposes.

AG-Graph Eigen Values

A naturally occurring subject in the realm of spectral theory of graph matrices is the one that we are discussing here.

Problem 1 Given a connected graph G with a rank of n ≥ 2, let M(G) be a graph matrix associated with the graph at hand and $k(1 \leq k \leq n),$ a positive integer is required. Graphs that have precisely k different M(G) -eigenvalues are characterised as having these characteristics.

This problem has been investigated for a number of matrices, including the adjacency matrix, the normalised Laplacian matrix, the distance matrix, and others, provided that the value of k is relatively small. In order to get further information, kindly consult the studies that were carried. There are a number of articles in the body of knowledge that investigate this subject for the matrices that have been described, particularly in situations when k is equal to or less than four. For instance, you might refer to the study that was carried, in addition to the references that are contained in these papers.

An inconsequential issue is the fact that nK1 is the sole complete graph that has just one AG-eigenvalue and that it's AG-spectrum is insignificant. This is a matter of minimal importance $\{0^{[n]}\}$. (Newman, M. E,2006).

Given the following well-known conclusion, there is a connection that can be made between the diameter of a graph and the number of distinct eigenvalues that it has.

Theorem 1. Let us entertain the notion that G is a size D linked graph. Brouwer and Haemers (2010) state that G has a minimum of D + 1 unique (adjacency) eigenvalues, a minimum of D + 1 distinct Laplace eigenvalues, and a minimum of D + 1 different signless Laplace eigenvalues. G also has unique eigenvalues that are at least D + 1.

As shown by the evidence that was offered in Brouwer and Haemers (2010), the aforementioned result is attainable by any nonnegative symmetric matrix $M = (M_{ij})_n$ indexed by the vertices of a graph G, in which $M_{ij} > 0$ if and only if $v_i \sim v_j$. As a result, the following corollary follows automatically.

Objectives Of The Study

1. To study on Relationships between structures and eigenvalues in spectral theory graphs
2. To study on taking into consideration the various spectrums of the Laplacian and the adjacency matrix

2. RESEARCH METHOD

As a result of expanding the technique, bounds on normalized eigengap differences are once again established, and it is shown that these bounds are, once again, in terms of the degree extremes of the graph. It has been shown that graph signal processing algorithms that have a lower degree of severe differences provide consistent outcomes. This is true regardless of the representation matrix that is used. The use of graph signal processing procedures that include a greater degree of severe disparities, on the other hand, has the potential to provide results that differ from those obtained via the use of various representation matrix options. To demonstrate the various implications that can be derived from signal processing methods utilizing the spectral clustering approach.

The affine transformations' motivation

As a result of suitably specified affine transformations, the three graph representation matrices for d-regular graphs have spectra that are totally linked. This is because D equals dI. The representation spectrum relation with regard to generic graphs

is an example of a non-linear representation. In situations where unequal eigenvalues in one spectrum correspond to repeated eigenvalues in another spectrum, one may discover evidence that the transformation between representation spectra is either non-existent or a polynomial of degree less than or equal to n. This evidence can be found by referring to Appendix A.

The process of establishing the polynomial mapping across spectra is one that requires a significant amount of computer effort, as shown in Appendix A. Furthermore, during these nonlinear alterations, significant spectral features such as the amplitude of the eigengap and the ordering of the eigenvalues are lost. However, it is important to keep in mind that the spectral clustering approach employs eigengaps in order to determine the cluster count of the network. An even more fundamental feature known as eigenvalue ordering is used in order to further cut down the selection of eigenvectors that are to be taken into consideration. For the purpose of performing a meaningful assessment of the impacts of representation matrix selection on data visualization, it is essential to maintain the eigenvalue ordering and the eigengap size that we have determined.

The purpose of this investigation is to restrict the eigenvalue differences that result from affine transformations across spectra that belong to generic graphs for the purpose of this research. The eigengap size and the ordering of the eigenvalues are both preserved by the use of Affine transformations. (Chen, X et al,2014)

Comparison of the representation spectrum based purely on the use of an additive constant or without going through the process of conversion is not meaningful. This is the outcome that occurs as a consequence of the fact that the eigenvalue orderings of the Laplacian matrices and the adjacency matrix are distinct as well. To restate the statement, the eigenvalue of the Laplacian that is the lowest is equivalent to the eigenvalue of the adjacency matrix that is the largest. Consequently, prior to comparing spectra, it would make sense to preprocess one of the spectra by using a multiplicative constant. An additional advantage of using an affine transformation is that it enhances the comparability of the spectra. This is accomplished by mapping the spectrum supports of the three representation matrices onto one another. It is important to scale in order to accomplish this mapping of spectral supports.

In order to facilitate the comparison of the three representation matrices, let the potential pairings X and Y serve as the representations. An affine transform, denoted by the symbol F(X), is defined as an affine transform of one of them. The next step is to investigate the bounds of the eigenvalue error.

$$|\lambda_i(\mathcal{F}(X)) - \lambda_i(Y)|,$$

Where λi(Y) is Y's eigenvalue.

The exact correlations that exist between the spectra of the three matrices for graphs that are d-regular may be retrieved via the manipulations that we do. In order to properly consider mappings between the representation spectra, it is necessary to investigate this baseline.

Contributions

Contributions made by this paper are as follows:

1. A structure that enables comparisons to be made between the representation matrices. The fundamental component of this system is an affine transformation, which provides a mapping from one matrix to another in spectral norm that is as close to being possible as possible. Given the universality of this approach, it is likely that it may be used for a variety of matrix comparison examples.
2. It is possible for us to assess the spectral disparity between the representation matrices on the eigenvalue and eigengap levels by making use of closed form limits.
3. A split of graphs created from the borders, according to their degree extremes, makes it feasible for us to conduct an insightful examination of our data.
4. This concept is based on the fact that lesser degree extreme differences will provide consistent inference, regardless of the representation matrix that is used, and that bigger degree extreme differences may lead to different findings from graph signal processing techniques.

Additionally, a summary of the characteristics of the adjustment that was applied is included with each comparison. By doing an analysis of the bound values across all graphs that are separated by their degree extremes, a proof of concept is shown, and the tightness of one of the limits is discovered on two of the cases; this is accomplished. In order to do this, the constraints are applied to three distinct instances of the model as well as graphs that are constructed from a social network. There is a straightforward method for determining the bound on the normalized eigengap differences for each and every pair of representation matrices that is provided. (Meesum, S. M et al,2016)

3. DATA ANALYSIS

The adjacency matrix, the normalized and unnormalized graph Laplacian matrices, and the adjacency matrix once again are

the three matrices that are often used for the purpose of representation of the structure of a graph. From this point forward, we are going to refer to these three matrices as representation matrices for the whole of the conversation. The study of graph topologies and the ways in which they might be used to modeling is now attracting a significant number of individuals. The spectrum features, also known as eigenvalues, of a representation matrix are often used in a particular investigation. In the case of the spectral clustering approach with graph wavelets, for example, there is only one representation matrix that may be used.

Limiting the individual difference of eigenvalues for A and L_{rw}

We are now moving on to the last of our three potential relationships: the spectrums of A and L_{rw}. Let L_{rw} have eigenvalues $\eta 1 \leq \cdots \leq \eta n$. Now, with $c_2, d_2 \in R$,

$$\begin{aligned} Aw &= \mu w \\ \iff (d_2 I - D + L)\, w &= (d_2 - \mu)\, w \\ \iff c_2\,(d_2 I - D + DL_{rw})\, w &= c_2\,(d_2 - \mu)\, w \\ \tilde{L}_{rw} w &= \eta(\tilde{L}_{rw}) w, \end{aligned} \quad \text{.....(1)}$$

Were

$$\begin{aligned} \tilde{L}_{rw} &\stackrel{\text{def}}{=} c_2\,(d_2 I - D + DL_{rw}) \\ &= c_2(d_2 I - A) \end{aligned} \quad \text{.....(2)}$$

$$\eta(\tilde{L}_{rw}) \stackrel{\text{def}}{=} c_2\,(d_2 - \mu)\,. \quad \text{.....(3)}$$

An affine transform of A is denoted by the notation L_{rw}, which has transform parameters c_2 and d_2. From a different perspective, one may consider it to be an additive and multiplicative perturbation of L_{rw}. When this occurs, the parameter choices and the affine transformation will coincide with one another.

$$A \to \mathcal{F}_3(A) \stackrel{\text{def}}{=} \tilde{L}_{rw} \approx L_{rw},$$

$F_3(A)$ is the notation used to refer to the affine transform in equation (2). It is important to keep in mind that since L_{rw} is not clearly defined for $d_{min} = 0$, we need d_{min} to be greater than 0.

We will choose the two free parameters, c_2 and d_2, in such a manner that the upper limit of the Eigen value differences is reduced to the greatest extent possible by doing so. Our first step is to

$$\left\|\tilde{L}_{rw} - L_{rw}\right\|_2 \geq \left|\eta_i(\tilde{L}_{rw}) - \eta_i\right| = |c_2(d_2 - \mu_i) - \eta_i|, \quad \text{.....(4)}$$

Regarding $i \in \{1, 2, \ldots, n\}$. Next,

$$\begin{aligned} \left\|\tilde{L}_{rw} - L_{rw}\right\|_2 &= \|c_2 d_2 I - c_2 D + (c_2 D - I)\, L_{rw}\|_2 \\ &\leq \|c_2 d_2 I - c_2 D\|_2 + 2\,\|c_2 D - I\|_2 \end{aligned} \quad \text{.....(5)}$$

$$\leq |c_2| \frac{d_{\max} - d_{\min}}{2}$$

$$+ 2 \max\left(|d_{\max} c_2 - 1|\,, |d_{\min} c_2 - 1|\right) \quad \text{.....(6)}$$

Given that the term being minimised is the same in both scenarios, we choose $d_2 = (d_{max} + d_{min})/2$ just as we did in (5). Since c_2 occurs in both terms, the choice is relevant. Appendix C provides details, demonstrating that the option $c2 = 2/(d_{max} + d_{min})$ indicates that

$$e\,(A, L_{rw}) \stackrel{\text{def}}{=} 3\,\frac{d_{\max} - d_{\min}}{d_{\max} + d_{\min}} \quad \text{.....(7)}$$

Due to the fact that the restriction is equal to zero for d-regular graphs, the spectra of L˜ rw and Lrw are identical to one another. The spectra of Lrw and A are connected by the required precise connection, as shown in equation (14), which

indicates that ηi = c2(d2− μi) = 1 − (μi/d). This is an evidence that the spectra of Lrw and A are linked.

The characteristics of transformations

We are going to investigate their characteristics and examine the spectra on a real data set in order to better comprehend the changes that occur in the spectra. f1, f2, and f3 are the names that will be used to refer to the transformations that are described in (3), (4), and (5), respectively.

The equations $d_1 = d_2 = (d_{max} + d_{min})/2$ and $c_1 = c_2 = 2/(d_{max} + d_{min})$ hold true here.

$$f_1(\mu) = d_1 - \mu; \quad \text{.....(8)}$$

$$f_2(\lambda) = c_1\lambda; \quad \text{.....(9)}$$

$$f_3(\mu) = c_2(d_2 - \mu) = 1 - c_2\mu \quad \text{.....(10)}$$

Example of Karate data

Since Zachary has provided us with a karate dataset, we immediately proceed to make the necessary adjustments to its spectra before continuing with the analysis. The karate dataset was obtained, and it includes information that pertains to the history of the sport as well. The "ZACHE" square matrix, which is a representation of the adjacency matrix of the social network and demonstrates the presence of interaction inside a university karate club, has a total of 34 entries. Our attention is focused on this matrix. This particular data set is often used as a benchmark in the academic literature due to the fact that it clusters so well.

The representation matrices A, L, and Lrw are shown in Figure 1. These matrices reflect the eigenvalues of Zachary's karate dataset both before and after the transformation. Following the modifications that were proposed, the eigenvalue spectra are now more straightforward to compare. The results shown in Figure 1(d) are used.

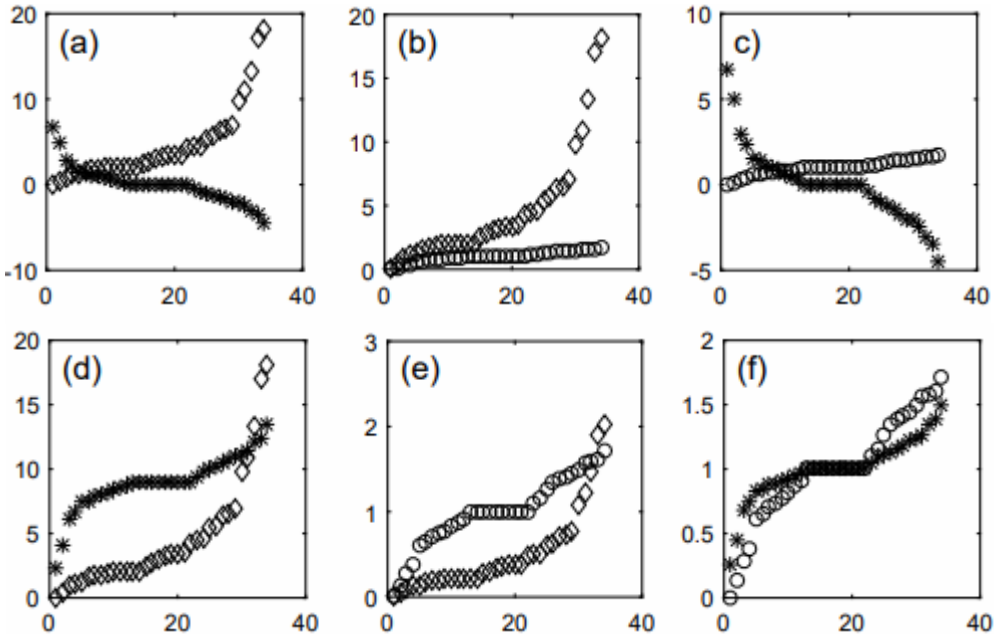

Figure 1. Results for Zachary's karate in the dataset. The first row displays the eigenvalues of A (stars) and L (diamonds) as μ and λ, respectively. The second row displays the eigenvalues of A and Lrw, as well as the eigenvalues of L (circles) and Lrw (stars). In the second row, we see the second eigenvalue spectrum after applying the modifications f1, f2, and f3.

It is clear from questions (e) and (f) that, after being transformed, each and every pair of spectra in the Karate dataset spans a range that is equivalent to one another. In spite of this, they do not align with one another, and it is feasible that they communicate distinct information about the underlying network.

Characteristics about transformation

f1, f2, and f3 are the three components that make up an affine transformation. It is important to note the following fundamental points with regard to generic affine transformations: Given a, b ∈ R and a 6= 0, g(x) = axe + b.

1. **Ordering:** The order is maintained:

$$\begin{aligned} & \mu_1 \geq \mu_2 \\ \Longleftrightarrow \; & a\mu_1 + b \geq a\mu_2 + b \\ \Longleftrightarrow \; & g(\mu_1) \geq g(\mu_2). \end{aligned}$$

2. **Eigengaps:** Following that, we provide evidence that eigengaps are maintained with regard to the spectral support. Let us operate under the assumption that the domain of g is represented by the interval [x1, x2]. Therefore, the image of g is the same as the expression [g(x1), g(x2)]. Eigengaps that have been normalised are then preserved:

$$\frac{\mu_1 - \mu_2}{x_2 - x_1} = \frac{a(\mu_1 - \mu_2) + b - b}{a(x_2 - x_1) + b - b} = \frac{g(\mu_1) - g(\mu_2)}{g(x_2) - g(x_1)}. \quad \ldots\ldots\ldots\ldots\ldots (11)$$

3. **Spectral support mapping:** In conclusion, we give the mappings that correspond to the spectrum supports of the different representation matrices. In order to get the spectral supports of the three representation matrices, it is possible to make use of Gershgorin's theorem.

$$f_1 : [-d_{\max}, d_{\max}] \rightarrow \left[-\frac{d_{\max} - d_{\min}}{2}, \frac{3d_{\max} + d_{\min}}{2} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$f_2 : [0, 2d_{\max}] \rightarrow \left[0, \frac{4d_{\max}}{d_{\max} + d_{\min}} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$f_3 : [-d_{\max}, d_{\max}] \rightarrow \left[-\frac{d_{\max} - d_{\min}}{d_{\max} + d_{\min}}, \frac{3d_{\max} + d_{\min}}{d_{\max} + d_{\min}} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

By selecting the transformation parameters in such a way that the transformation properly aligns with the spectral support of the target matrix, it is feasible to customize the transformation. On the other hand, this results in a substantial limitation on the degree to which the eigenvalues diverge from one another. As an alternative, we choose our transformation parameters on the basis of the bound value, how they deal with the ordering of eigenvalues, and how closely they conserve relative eigengaps. This is because the mapped supports are not of great significance.

Table 1 Evaluate the limits on the differences in eigenvalues of a, l, and lrw. Displayed as (e (a, l), e (l, lrw), e (a, lrw)) are the bound values. The following labels are used to the six distinct areas: e (a, l) = e (l, lrw) = e (a, lrw) = 0 as bold and underlined, e (a, l) < e (l, lrw) < e (a, lrw) as bold, e (a, l) = e (l, lrw) < e (a, lrw) as underlined, e (l, lrw) < e (a, l) < e (a, lrw) as t e l e t y p e d, e (l, lrw) < e (a, l) = e (a, lrw) as italic and finally e (l, lrw) < e (a, lrw) < e (a, l) in normal font.

D_{min}							
		0	1	2	3	4	5
D_{max}	1	(0.5, ·, ·)	(0, 0, 0)	-	-	-	-
	2	(1, ·, ·)	(0.5, 0.67, 1)	(0, 0, 0)	-	-	-
	3	(1.5, ·, ·)	(1, 1, 1.5)	(0.5, 0.4, 0.6)	(0, 0, 0)	-	-
	4	(2, ·, ·)	(1.5, 1.2, 1.8)	(1, 0.67, 1)	(0.5, 0.29, 0.43)	(0, 0, 0)	-
	5	(2.5, ·, ·)	(2, 1.33, 2)	(1.5, 0.86, 1.29)	(1, 0.5, 0.75)	(0.5, 0.22,0.33)	(0, 0, 0)
	6	(3, ·, ·)	(2.5, 1.43, 2.14)	(2, 1, 1.5)	(1.5, 0.67, 1)	(1, 0.4, 0.6)	(0.5, 0.18, 0.27)
	7	(3, ·, ·)	(3, 1.5, 2.25)	(2.5, 1.11, 1.67)	(2, 0.8, 1.2)	(1.5, 0.55, 0.82)	(1, 0.33, 0.5)

Partitioning

For the purpose of highlighting the class to which the boundaries apply, we will now demonstrate the class of unweighted graphs that is characterized by two degree extremes, dmin and dmax. The information will assist us in gaining an understanding of the monotonicity of the three borders in reference to Table I. The class of graphs representing the situation in which dmax equals k and dmin equals j shall be denoted by the symbol Cj,k.

We will study the class of graphs C,2, which is often referred to as the class of graphs to which the limitations in the second row apply. This is done for the purpose of demonstrating anything. The distribution of C,2 is shown in Figure 2, which also provides an overview of the constituents that constitute each class. As a result of the fact that there is total flexibility in the size of all classes Cj,k, we only exhibit a limited number of samples at random.

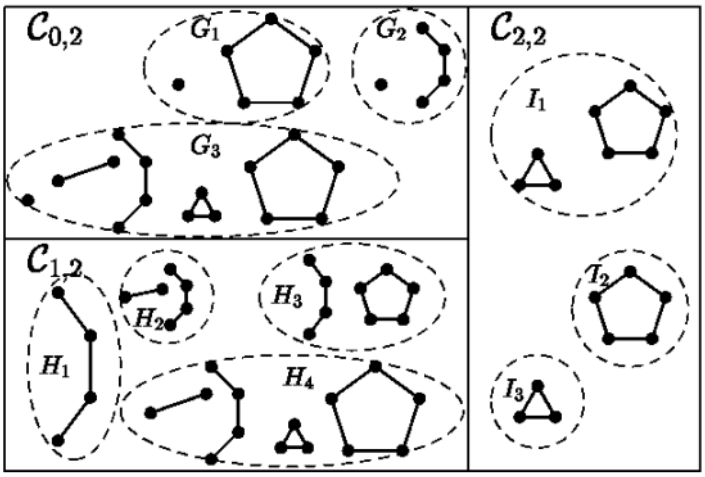

Figure 2 This diagram illustrates the division of C,2 = C0,2∧C1,2∧C2,2. The two-regular graph classes C0,2 (consisting of all graphs with dmin = 0 and dmax = 2), C1,2 (including elements for which dmin = 1 and dmax = 2), and C2,2 are shown with examples.

Incorporating and removing components that are related

At this point, we will concentrate on a specific graph alteration, which is the addition and removal of related components. Because of this, we are able to organise the graph spectra and, as a result, appreciate the monotonicity that is shown in the borders.

A single disconnected vertex may be added to G3 in order to get G3 from H4, and the linked components, line, and 2-complete component can be added in order to obtain H4 from I1. As is evident by looking at Figure 2. Generally speaking, a network in Cj,k may be constructed by using a graph in Cj+1,k as the starting point. An extra connected component that includes at least one vertex with degree j and all vertices with degrees between j and k is one technique to accomplish this goal. This may be accomplished by augmenting the graph with related components. Any value of j that is between k minus one, where k is an element of the set N, is considered to be subject to this statement.

Analysis of monotonicity in bounds

There was a decrease in the limit size that we observed when the indices that corresponded to dmin grew. At the same time as the indices that correspond to d_{max} are increasing, this suggests that the bound size is expanding. Considering that increasing the support of the degree distribution ought to result in an increase in the spectral support and, therefore, the limits, this makes perfect sense. There is a probability distribution known as the degree distribution that is used in order to sample the vertex degrees of a graph.

From a graph in Cj,k, we are able to construct any network in Cj,k+1 by linking nodes that have vertex degrees that are larger than or equal to j and less than or equal to k + 1. It is for this reason why we are here. Additionally, at least one node must achieve a degree of k plus one or higher.

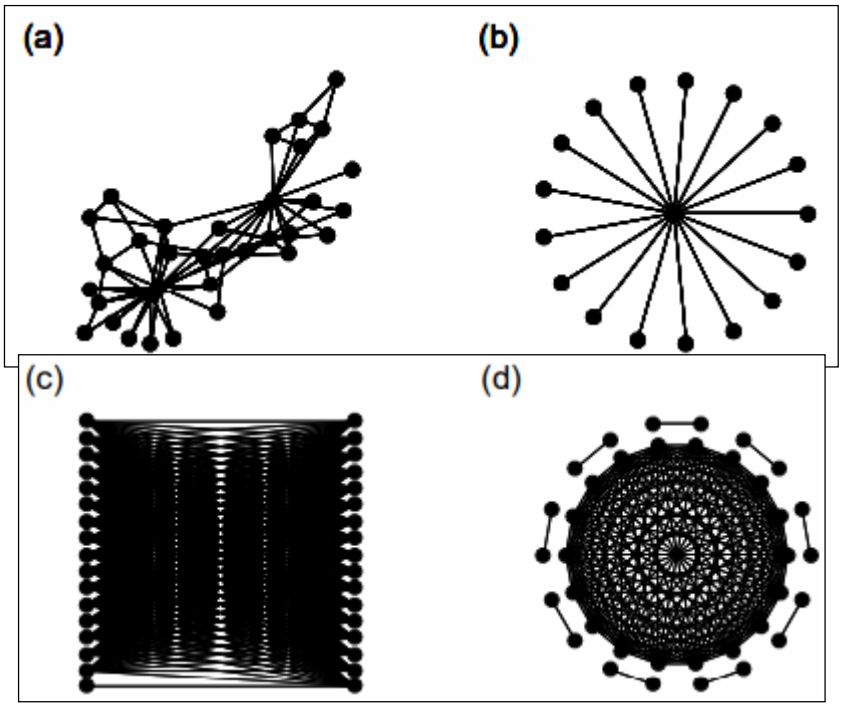

Figure 3. (a) The karate data set's graph. (b) Graph A, an 18-node star graph. (c) A bipartite graph with a degree distribution, called graph B. (d) The graph C, which has one 18-complete component and nine 2-complete components.

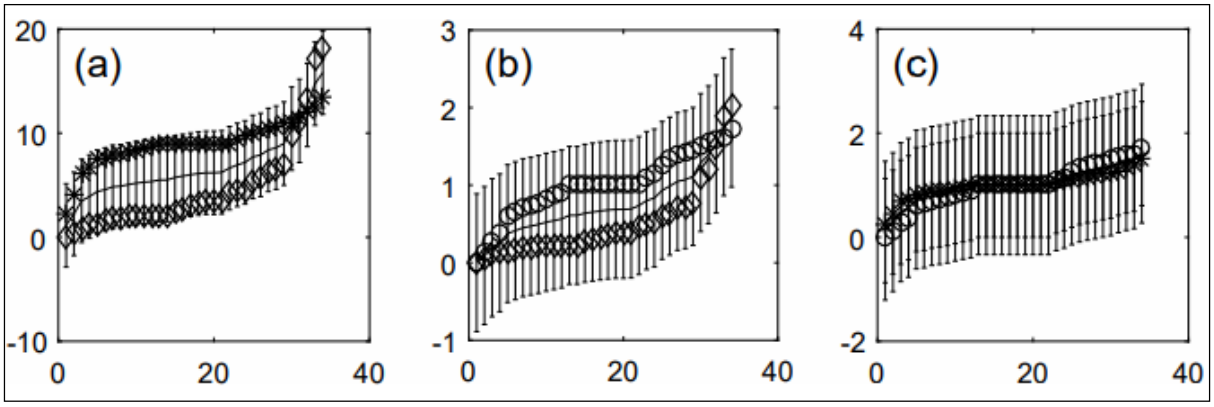

Figure 4. The karate eigenvalues are constrained by certain eigenvalue restrictions. The bound e(A, L) is shown in plot (a) along the intervals. The Laplacian λ is represented by diamonds, and the converted eigenvalues of the adjacency matrix f1(μ) are represented by stars. (b) The intervals display the limit e(L, Lrw); the circles correspond to the eigenvalues of the normalized graph Laplacian η, and the diamonds to the converted Laplacian eigenvalues f2(λ). Refer to Remark 3 for more details; figure (c) demonstrates that the restriction e(A, Lrw) is equivalent to the outer boundaries interval (e ′ (A, Lrw)). In the picture, the stars stand for the averaged Laplacian eigenvalues η and the converted adjacency eigenvalues f3(μ).

Because the karate data set provides the structure of the graph, the quality and tightness of the borders are less significant than the structure of the graph itself. As a result of the fact that the three constraints e(A, L), e(L, Lrw), and e(A, Lrw) only apply to full classes Cj,k concurrently, we can only hope to accomplish tightness on these classes as a whole and not on each individual member of them. As a result, the limits are being fulfilled for specific components in Cj,k.

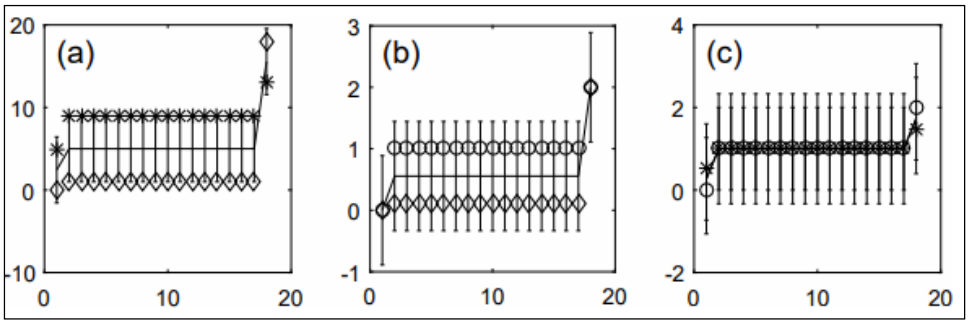

Figure 5 For graph A, a star with 18 nodes, spectra and modified spectra along with their bounds are shown. The degree sequence is also included.

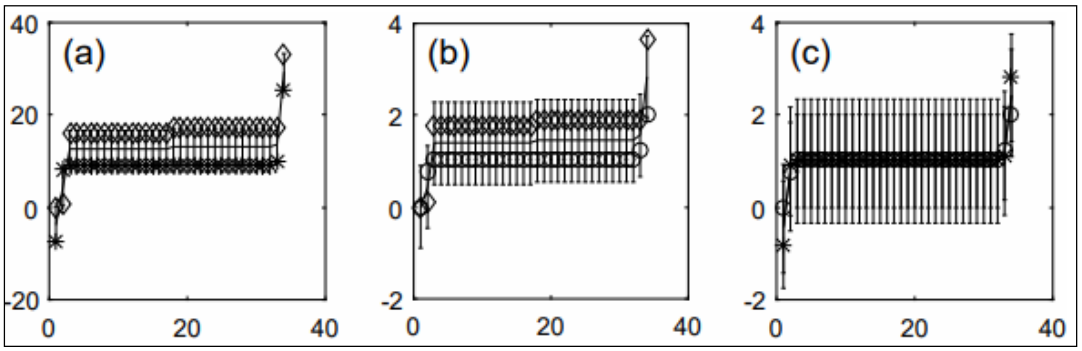

Figure 6. Graph B is a bipartite structure with 34 nodes and a degree sequence; these are its spectra, changed spectra, and limits.

Particularly interesting is the fact that Fig. 5(a) demonstrates that the limit e(A, L) is quite close: for sixteen of the eighteen eigenvalues, the spectra of A and L have the greatest distance on the individual eigenvalue level.

Limits on normalised disparities in eigengaps

Let's have a look at the eigengaps that correspond to the three representation matrices in this particular instance. We are going to compare and contrast the limits on eigengaps that are normalised by the spectral support with the restrictions on eigengaps that are generated from eigenvalues that have been changed by f1, f2, and f3. In addition to this, we investigate the criteria on the spectra that are responsible for the narrowing of the eigengap limits that are produced.

A and L's normalised eigengap difference

Mi = μi − μi+1 is the representation of the i th eigengap of A, while Li = λi+1 − λi is the representation of the i th eigengap of L, where i belongs to the set of numbers {1, 2,..., n − 1}. To refresh your memory, the spectral supports of A are [− dmax, dmax], whereas the spectral supports of L are [0, 2dmax]. As a result, the support length is written as $\ell(\mu) = \ell(\lambda) =$ 2dmax.

As a result of (8) and (12),

$$\left|\frac{f_1(\mu_{i+1}) - f_1(\mu_i)}{\ell(f_1(\mu))} - \frac{\mathcal{L}_i}{\ell(\lambda)}\right| = \left|\frac{\mathcal{M}_i}{2d_{\max}} - \frac{\mathcal{L}_i}{2d_{\max}}\right|$$

$$= \left|\frac{\mathcal{M}_i}{\ell(\mu)} - \frac{\mathcal{L}_i}{\ell(\lambda)}\right|,$$

as one would expect from (11). Additionally, because ℓ(μ) = ℓ(λ), normalising the two boundaries independently is the same as normalising the whole difference by a single number. Where the eigenvalue supports' lengths vary, this won't be the case.

Consequently, the bound on the normalized eigengap difference, denoted by the expression g(A, L), has the following syntax:

$$\left|\frac{\mathcal{M}_i}{2d_{\max}} - \frac{\mathcal{L}_i}{2d_{\max}}\right| = \frac{1}{2d_{\max}}|\mu_i - \mu_{i+1} - \lambda_{i+1} + \lambda_i|$$
$$= \frac{1}{2d_{\max}}\left|\mu_i - \frac{d_{\max}+d_{\min}}{2} + \lambda_i - \mu_{i+1} + \frac{d_{\max}+d_{\min}}{2} - \lambda_{i+1}\right|$$

$$= \frac{1}{2d_{\max}}|[\lambda_i - f_1(\mu_i)] + [f_1(\mu_{i+1}) - \lambda_{i+1}]| \quad \text{.....(15)}$$

$$\leq \frac{1}{2d_{\max}} 2e(A, L) \quad \text{.....(16)}$$

$$= \frac{d_{\max} - d_{\min}}{2d_{\max}} \stackrel{\text{def}}{=} g(A, L)$$

Using the triangle inequality, we are able to go from the number 15 to the number 16. The conclusion that can be drawn from Equation (15) is that the eigengaps constraint is considered to be tight if and only if the mapped eigenvalue differences, denoted as λi − f1(μi) and λi+1 − f1(μi+1), possess non-zero signs and are located at the extremes of the limit e(A, L).

L and L_{rw}'s normalised eigengap difference

In the next step, we will set a limit on the normalized eigengap difference between L and Lrw, which will be represented by the symbol g(L, Lrw). The i th eigengap of Lrw is represented by the equation Ni = ηi+1− ηi, where i is a member of the set comprised of elements ranging from 1 to n minus 1. Remember that the value of ℓ(ηi) is equal to 2 whenever you are working with Lrw.

After normalizing the transformed eigenvalues according to their support, the first thing that we are going to do is compare the normalized and untransformed spectra compared to one another.

$$\left|\frac{f_2(\lambda_{i+1}) - f_2(\lambda_i)}{\ell(f_2(\lambda))} - \frac{\mathcal{N}_i}{\ell(\eta)}\right| = \left|\frac{c_1}{\frac{2}{d_{\max}+d_{\min}}}\frac{[\lambda_{i+1} - \lambda_i]}{2d_{\max}} - \frac{\mathcal{N}_i}{2}\right|$$
$$= \left|\frac{\mathcal{L}_i}{\ell(\lambda)} - \frac{\mathcal{N}_i}{\ell(\eta)}\right|,$$

to the extent that one would expect from (13). Following that, we get the difference due to the normalised eigengap:

$$\left|\frac{\mathcal{L}_i}{2d_{\max}} - \frac{\mathcal{N}_i}{2}\right| = \frac{1}{2}\left|\frac{1}{d_{\max}}(\lambda_{i+1} - \lambda_i) + \eta_i - \eta_{i+1}\right| \quad \text{.....(17)}$$

$$\leq \frac{1}{2}\left[\left|\frac{1}{d_{\max}}\lambda_{i+1} - \eta_{i+1}\right| + \left|\frac{1}{d_{\max}}\lambda_i - \eta_i\right|\right]$$

A and $Lr_{w's}$ normalised eigengap difference

we will get the constraint on the normalised eigengaps of A and Lrw, which will be denoted by the expression g(A, Lrw). Both (13) and (16) make it very clear that (14) is correct:

$$\left|\frac{f_3(\mu_{i+1})-f_3(\mu_i)}{\ell(f_3(\mu))}-\frac{\mathcal{N}_i}{\ell(\eta)}\right| = \left|\frac{c_2}{\frac{2}{d_{\max}+d_{\min}}}\frac{\mu_i-\mu_{i+1}}{2d_{\max}}-\frac{\mathcal{N}_i}{2}\right|$$
$$= \left|\frac{\mathcal{M}_i}{\ell(\mu)}-\frac{\mathcal{N}_i}{\ell(\eta)}\right|.$$

In light of this, we proceed by establishing the restriction that applies to the normalised eigengaps.

$$\left|\frac{\mathcal{M}_i}{2d_{\max}}-\frac{\mathcal{N}_i}{2}\right| = \frac{1}{2}\left|\frac{1}{d_{\max}}(\mu_i-\mu_{i+1}+d_2-d_2)+\eta_i-\eta_{i+1}\right|$$
$$\leq \frac{1}{2}\left[\left|\frac{1}{d_{\max}}(d_2-\mu_{i+1})-\eta_{i+1}\right| + \left|\frac{1}{d_{\max}}(d_2-\mu_i)-\eta_i\right|\right],$$

thanks to the fact that the triangle is not equal. Since the scaling factor in equation (16) does not have any impact on the d2 optimisation, we are able to verify that the limit is minimised by the equation d2 = (d_{max} + dmin)/2. After that, a comparison with (15) reveals that these eigenvalue differences may be limited by adding 1/d_{max} for c2 and then into (17) to reach the desired result

$$\left|\frac{\mathcal{M}_i}{2d_{\max}}-\frac{\mathcal{N}_i}{2}\right| \leq \frac{d_{\max}-d_{\min}}{2d_{\max}}+2\max\left(0,\left|\frac{d_{\min}}{d_{\max}}-1\right|\right)$$
$$= \frac{5}{2}\frac{(d_{\max}-d_{\min})}{d_{\max}} \stackrel{\text{def}}{=} g(A, L_{rw}).$$

In the same manner as Sections IX-A and IX-B, the g(A, Lrw) function is a tight limit when there is a maximum crossover in the normalised spectra, with the equation φ(μ) = (d2 − μ)/d_{max} applying.

4. CONCLUSION

Interactions between the adjacency matrix A, the unnormalized graph Laplacian L, and the normalized graph Laplacian Lrw revealed spectrum patterns that are indicative of generic graphs. It turned out that these patterns didn't match up. It was discovered that the degree extreme difference (dmax − dmin) acts as a linear upper limit for the inaccuracy when calculating one's eigenvalue spectra using an affine transformation of the other matrix's spectrum. All three distinct permutations of representation matrices had this settled. After the affine transformations were considered, we found that the lowest Laplacian eigenvalues coincide with the greatest adjacency eigenvalues. Our proof of monotonicity in the limits relies on the uniformity of the borders, which we achieve by dividing the class of graphs according to their degree extremes and by considering the addition or removal of linked components from the graph. In the other direction, graph signal processing methods may provide contradictory inferences owing to a significant degree of extreme difference when a variety of representation matrix options are employed. On the other hand, regardless of the representation matrix that is used, conclusions that are based on aberrations of a smaller degree of severity will remain consistent. This was seen in instances that had dramatically different degrees of intensity and severity. Zachary was able to generate karate and model graphs with the use of the spectral clustering approach. These graphs let one to see how signal processing methods produced outcomes that were radically different from one another. As a part of our research, we would want to have a better understanding of the significance of appropriate representation matrix selection for applications involving graph signal processing.

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