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Existence of Solutions For Higher Order Impulsive Integro-Differential Equations With Antiperiodic Boundary Value Problems On Time Scales

Arunkumar D[1], Kannan K[2], Anitha K[3], Sujatha B[4], Gnanavel M G[5]

[1]Assistant Professor, Hindusthan College of Engineering and Technology, Coimbatore.
Email ID: arunkumar.maths@hicet.ac.in
[2]Professor, Hindusthan College of Engineering and Technology, Coimbatore
[3]Associate Professor, Hindusthan College of Engineering and Technology, Coimbatore
[4]Assistant Professor, Hindusthan College of Engineering and Technology, Coimbatore
[5]Assistant Professor, Hindusthan College of Engineering and Technology, Coimbatore
Email ID: kannan.maths@hicet.ac.in

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ABSTRACT
In this paper, we investigate the higher order impulsive integro-differential equations with anti periodic boundary value problems on time scales. The Contraction mapping principle and Leray Schauder's fixed point theorem are used to determine sufficient conditions. Finally, we present an example to demonstrate the main results.

Keywords: *Impulsive, Integro-differential equations, Antiperiodic boundary conditions, Timescale calculus*

1. INTRODUCTION

The theory of impulsive differential equations affords adequate mathematical models for describing evolution processes characterized by the combination of continuous and jump fluctuations in their state. In recent years, impulsive differential equations have become a significant area of exploration that serves the needs of modern technology, engineering, economics, physics, and so on. Impulsive differential equations with several boundary conditions are studied by many authors in[6,11,12]. At the same time, the theory of boundary value problems with antiperiodic boundary conditions for differential equations arises in different areas of applied mathematics and physics. But the corresponding theory for impulsive integro-differential equations on time scales is yet to be developed. Integro-differential equations are typical of those processes where a function at each point is not determined by its value near the point, but also depends on the function distribution all over the domain. Impulsive Integro-differential equations with boundary conditions also studied by several researchers in [2,5,10,12]. Periodic and Antiperiodic boundary conditions with different type of equations are established by the researchers in [5,9,10].

The calculus of time scales was initiated by Stefan Hilger, in order to create a theory that can unify discrete and continuous analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Higher order Differential equations are studied by the several authors in [1,3,4,7,13,14]. Motivated by above, this study consider the following n^{th} order impulsive integro differential equations with antiperiodic boundary conditions on time scales:

$$\begin{aligned} &x^{\Delta^n}(t) = f\Big(t, x(t), x^{\Delta}(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s\Big),\\ &\Delta x^{\Delta^{(n-1)}}|_{t=t_k} = I_k(x(t_k)), t \in [0,1]_{\mathbb{T}}, t \neq t_k, k = 1,2,\cdots,m, \qquad (1)\\ &x^{\Delta^{(i)}}(0) = -x^{\Delta^{(i)}}(\sigma(T)), i = 0,1,2,\cdots,n-1. \end{aligned}$$

Where $f: [0,T]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$, $g: [0,T]_{\mathbb{T}} \times [0,T]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ and $I_k: \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $\Delta x^{\Delta^{(n-1)}}|_{t=t_k} = x^{\Delta^{(n-1)}}(t_k^+) - x^{\Delta^{(n-1)}}(t_k^-)$. Here $x^{\Delta^{(n-1)}}(t_k^+), x^{\Delta^{(n-1)}}(t_k^-)$ represent the right hand limit and left hand limit of $x^{\Delta^{(n-1)}}(t)$ at $t = t_k, k = 1,2,\cdots,m, m$ is a fixed positive integer with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1$.

The article is organized as follows. In Section 2, we present some lemmas and concepts to prove the main results. Section 3 contains the main results of the paper. In section 4, an example is given to illustrate how the main results work.

2. PRELIMINARIES

In order to define the solution of problem (1), we will consider the following space.

Let $J' = J\backslash\{t_1, t_2, \cdots, t_n\}$ and

$$\mathrm{PC}^{n-1}[0,1]_{\mathbb{T}} = \left\{x \in C[0,1]_{\mathbb{T}}: x^{\Delta^{(n-1)}}|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}), x^{\Delta^{(n-1)}}(t_k^-) = x^{\Delta^{(n-1)}}(t_k), k = 1,2,\cdots, m\right\}.$$

Then $PC^{n-1}[0,1]_{\mathbb{T}}$ is a real Banach space with norm

$$||x||_{PC^{n-1}} = \max\left\{||x||_\infty, \left|\left|x^{\Delta}\right|\right|_\infty, \left|\left|x^{\Delta^2}\right|\right|_\infty, \cdots, \left|\left|x^{\Delta^{n-1}}\right|\right|_\infty\right\},$$

Where $\left|\left|x^{\Delta^{n-1}}\right|\right|_\infty = \sup_{t\in J} |x^{\Delta^{n-1}}(t)|, n = 1,2, \cdots.$

2.1 Definition

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. And the forward and backward jump operators $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ are defined by

$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$

respectively. The point $t \in \mathbb{T}$ is the called left dense, left scattered, right dense or right scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t$ or $\sigma(t) > t$ respectively. Points that are right dense and left dense at the same time are called dense.

If $\mathbb{T}$ has a left scattered maximum m, define $\mathbb{T}^k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^k = T$. The symbols $[a,b], [a,b)$ and so on, denote time scales intervals, for example,

$$[a,b] = \{t \in \mathbb{T} : a \le t \le b\},$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

2.2 Definition

A vector function $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous provided that it is continuous at each right dense point in $\mathbb{T}$ and has a left-sided limit at each left dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted in this paper by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

2.3 Definition

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of $t (i.e., \mathcal{U} = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0)$ such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\right| \le \epsilon|\sigma(t) - s|, \forall\, s \in \mathcal{U}.$$

We call $f^\Delta(t)$ the delta (or Higher) derivative of f at t.

2.4 Definition

If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

2.5 Lemma

Assume that $f, g: \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and α, β are two constants, we have:

1. if $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then $(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t)$,
2. if f^Δ exists, then f is continuous at t.

2.6 Lemma

If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$ and $f, g \in C_{rd}$ then:

1. $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$

2. $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$

3. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$

4. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$

5. $\int_a^a f(t)\Delta t = 0$

2.7 Theorem

Let E be a real Banach space, $A : E \to E$ is a completely continuous operator. If the set $\{x : x \in E, x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then A has at least one fixed point in the closed ball $T \in E$, where

$$T = \{x : x \in E, \| x \| \leq R\}, R = sup\{\| x \|: x = \lambda Ax, 0 < \lambda < 1\}.$$

2.8 Lemma

For any $h(t) \in PC([0, T], \mathbb{R}^n)$, $x(t)$ solves

$$x^{\Delta^{(n)}}(t) = h(t),$$
$$\Delta x^{\Delta^{(n-1)}}|_{t=t_k} = I_k(x(t_k)), t \in [0,1]_{\mathbb{T}}, t \neq t_k, k = 1, 2, \cdots, m, \qquad (2)$$
$$x^{\Delta^{(i)}}(0) = x^{\Delta^{(i)}}(\sigma(T)), i = 0, 1, 2, \cdots, n - 1.$$

if and only if $x(t)$ is the solution of integral equation

$$x(t) = \int_0^{\sigma(T)} G(t, \sigma(s))h(s)\Delta s - \sum_{0<t_k<t} G(t, t_k)I_k(x(t_k)) \qquad (3)$$

Where

$$G(t,s) = \begin{cases} \frac{\sigma(T)}{4}\frac{(t-s)^{n-2}}{(n-2)!} + \left(\frac{t-s}{2}\right)^{n-1} - \frac{(t-s)^{n-1}}{(n-1)!} & 0 \leq s \leq t \leq \sigma(T), \\ \frac{\sigma(T)}{4}\frac{(t-s)^{n-2}}{(n-2)!} + \left(\frac{t-s}{2}\right)^{n-1} & 0 \leq t \leq s \leq \sigma(T), \end{cases}$$

Proof: Assume $x(t)$ is a solution of (2). Then by integrating $x^{\Delta^{(n)}}(t) = h(t), t \neq t_k, (k = 1,2, \cdots, m)$ step by step from 0 to t, we have

$$x^{\Delta^{(n-1)}}(t) = x^{\Delta^{(n-1)}}(0) - \int_0^t h(s)\Delta s + \sum_{0<t_k<t} I_k(x(t_k)). \qquad (4)$$

Integrating the equation (4) again step by step from 0 to t, we get

$$x^{\Delta^{(n-2)}}(t) = x^{\Delta^{(n-2)}}(0) + x^{\Delta^{(n-1)}}(0)t - \int_0^t (t - \sigma(s))h(s)\Delta s + \sum_{0<t_k<t} I_k(x(t_k))(t - t_k). \qquad (5)$$

By using Antiperiodic boundary conditions in (4), we get

$$x^{\Delta^{(n-1)}}(0) = \frac{1}{2}\left[\int_0^{\sigma(T)} h(s)\Delta s - \sum_{0<t_k<t} I_k(x(t_k))\right]. \qquad (6)$$

Similarly, by using Antiperiodic boundary conditions in (5), we get

$$x^{\Delta^{(n-2)}}(0) = \frac{1}{2}\left[-x^{\Delta^{(n-1)}}(0)\big(\sigma(T)\big) + \int_0^{\sigma(T)} \big(\sigma(T)-\sigma(s)\big)h(s)\Delta s - \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)(\sigma(\mathrm{T})-\mathrm{t_k}).\right] \quad (7)$$

Substituting (6) and (7) in (5), we get

$$x^{\Delta^{(n-2)}}(t) = \int_0^{\sigma(T)} \left[\frac{\sigma(T)}{4} - \frac{\sigma(s)}{4} + \frac{t}{2}\right] h(s)\Delta s - \int_0^t \big(t-\sigma(s)\big)h(s)\Delta s$$
$$- \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\left[\frac{\sigma(T)}{4} - \frac{t_k}{4} + \frac{t}{2}\right] + \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)(\mathrm{t}-\mathrm{t_k}). \quad (8)$$

In the same way, we get

$$x^{\Delta^{(n-3)}}(t) = \int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\big(t-\sigma(s)\big) + \left(\frac{t-\sigma(s)}{2}\right)^2\right] h(s)\Delta s - \int_0^t \frac{\big(t-\sigma(s)\big)^2}{2} h(s)\Delta s$$
$$- \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\left[\left(\frac{\sigma(T)}{4}\right)(t-t_k) + \left(\frac{t-t_k}{2}\right)^2\right] + \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\frac{(\mathrm{t}-\mathrm{t_k})^2}{2}.$$

Similarly, we get

$$x(t) = \int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\frac{\big(t-\sigma(s)\big)^{n-2}}{(n-2)!} + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right] h(s)\Delta s - \int_0^t \frac{\big(t-\sigma(s)\big)^{n-1}}{(n-1)!} h(s)\Delta s$$
$$- \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\frac{(\mathrm{t}-\mathrm{t_k})^{\mathrm{n}-1}}{(\mathrm{n}-1)!}.$$

If we choose $G(t,s)$, then (3) is true.

Conversely, if we differerentiate (3) and by using boundary conditions, we get

$$x^{\Delta^{(n)}}(t) = h(t)$$

So $\Delta x^{\Delta^{(n-1)}}|_{t=t_k} = I_k\big(x(t_k)\big),, k = 1,2,\cdots,m$, and it is easy to verify that $x^{\Delta^{(i)}}(0) = x^{\Delta^{(i)}}(\sigma(T)), i = 0,1,2,\cdots,n-1$

and the lemma is proved.

3. MAIN RESULTS

In view of Lemma2.8, we define a fixed point problem related to problem (1) as $Fx = x$, where $F : C([0,T]_{\mathbb{T}},\mathbb{R}^n) \to C([0,T]_{\mathbb{T}},\mathbb{R}^n)$ is defined by

$$(Fx)(t) = \int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\frac{\big(t-\sigma(s)\big)^{n-2}}{(n-2)!}\right.$$
$$\left. + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right] \mathrm{f}\left(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}^{\Delta}(\mathrm{t}),\cdots,\mathrm{x}^{\Delta^{\mathrm{n}-1}}(\mathrm{t}),\int_0^{\mathrm{t}} \mathrm{g}(\mathrm{t},\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}^{\Delta}(\mathrm{s}),\cdots,\mathrm{x}^{\Delta^{\mathrm{n}-1}}(\mathrm{s}))\Delta \mathrm{s}\right)\Delta s$$
$$- \int_0^t \frac{\big(t-\sigma(s)\big)^{n-1}}{(n-1)!} \mathrm{f}\left(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}^{\Delta}(\mathrm{t}),\cdots,\mathrm{x}^{\Delta^{\mathrm{n}-1}}(\mathrm{t}),\int_0^{\mathrm{t}} \mathrm{g}(\mathrm{t},\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}^{\Delta}(\mathrm{s}),\cdots,\mathrm{x}^{\Delta^{\mathrm{n}-1}}(\mathrm{s}))\Delta \mathrm{s}\right)\Delta s$$
$$- \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t} \mathrm{I_k}\big(\mathrm{x}(\mathrm{t_k})\big)\frac{(\mathrm{t}-\mathrm{t_k})^{\mathrm{n}-1}}{(\mathrm{n}-1)!}.$$

We give the following hypothesis:

(H_1): For the subsequent analysis, we define $||x||_{PC^{n-1}} = \max_{t\in[0,T]_{\mathbb{T}}}\left\{||x||, ||x^{\Delta}||, \cdots, ||x^{\Delta^{n-1}}||\right\}$ and

$$\gamma = \sup_{t\in[0,T]_{\mathbb{T}}}\left\{\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!} + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\Delta s - \int_0^t \frac{(t-\sigma(s))^{n-1}}{(n-1)!}\Delta s\right\},$$

$$\beta = \sup_{t\in[0,T]_{\mathbb{T}}}\left\{-\sum_{0<t_k<t} I_k(x(t_k))\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t} I_k(x(t_k))\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}.$$

(H_2): The functions f, g and I_k are continuous.

(H_3): The functions f, g and I_k satisfying the Lipschitz condition:

$$\left|f(t,x,x^{\Delta},x^{\Delta^2},\cdots,x^{\Delta^{n-1}}) - f(t,y,y^{\Delta},y^{\Delta^2},\cdots,y^{\Delta^{n-1}})\right| \le L\left[|x-y| + |x^{\Delta}-y^{\Delta}| + \cdots + |x^{\Delta^{n-1}} - y^{\Delta^{n-1}}|\right],$$

$$\left|g(t,s,x,x^{\Delta},x^{\Delta^2},\cdots,x^{\Delta^{n-1}}) - g(t,s,y,y^{\Delta},y^{\Delta^2},\cdots,y^{\Delta^{n-1}})\right| \le M\left[|x-y| + |x^{\Delta}-y^{\Delta}| + \cdots + |x^{\Delta^{n-1}} - y^{\Delta^{n-1}}|\right],$$

$$\left|I_k(x(t_k)) - I_k(y(t_k))\right| \le Q|x-y|, L, M, Q > 0, \forall x, y \in \mathbb{R}^n, t \in [0,T]_{\mathbb{T}}.$$

(H_4): There exist a positive constant N, R such that $\left|f\left(t, x(t), x^{\Delta}(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s\right)\right| \le N,$

$\left|I_k(x(t_k))\right| \le R$, for each $t \in [0,T]_{\mathbb{T}}$ and all $x \in \mathbb{R}^n$.

3.1 Theorem

Assume that (H_2) and (H_3) holds. Then the boundary value problem (1) has a unique solution if $L\gamma + LM\gamma + Q\beta < 1$, where γ and β given by (10).

Proof: In the first step, we show that $FB_c \to B_c$, Where F is the operator defined by (14) and $B_c = \left\{x \in C([0,T]_{\mathbb{T}}, \mathbb{R}^n): ||x||_{PC_{n-1}} \le C\right\}$ with $c \ge \frac{d\gamma + e\beta}{1 - L\gamma - LM\gamma - Q\beta}$, $d = \sup_{t\in[0,T]_{\mathbb{T}}} |f(t,0)|$ and $e = \sup_{t\in[0,T]_{\mathbb{T}}} \left|I_k(x(0))\right|$.

Using $f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s) = |f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s) - f(t,0) + f(t,0)|$

$$\le |f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s) - f(t,0)| + |f(t,0)|$$

$$\le L\left[||x|| + ||x^{\Delta}|| + \cdots + ||x^{\Delta^{n-1}}|| + M\left(||x|| + ||x^{\Delta}|| + \cdots + ||x^{\Delta^{n-1}}||\right)\right] + d$$

$$\le L||x||_{PC_{n-1}} + LM||x||_{PC_{n-1}} + d \le Lc + LMc + d \text{ and}$$

$I_k(x(t_k)) = I_k(x(t_k)) + I_k(x(0)) - I_k(x(0)) \le \left|I_k(x(t_k)) - I_k(x(0))\right| + \left|I_k(x(0))\right| \le Q||x||_{PC_{n-1}} + e \le Qc + e$ for any $x \in B_c$, $t \in [0,T]_{\mathbb{T}}$, we obtain

$$|(Fx)(t)| = \sup_{t\in[0,T]_{\mathbb{T}}}\left\{\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!} + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\left|f\left(t,x(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s\right)\right|\Delta s \right.$$
$$- \int_0^t \frac{(t-\sigma(s))^{n-1}}{(n-1)!}\left|f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s)\right|\Delta s$$
$$\left. - \sum_{0<t_k<t}|I_k(x(t_k))|\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t}|I_k(x(t_k))|\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$
$$\le (Lc + LMc + d)\gamma + (Qc + e)\beta \le c,$$

which implies that $||Fx|| \le c$. In consequence, it follows that $FB_c \subset B_c$.Next for $x, y \in C([0,T]_{\mathbb{T}}, \mathbb{R}^n)$ and for each $t \in [0,T]_{\mathbb{T}}$, we have that

$|(Fx)(t)-(Fy)(t)|$

$$\leq \sup_{t\in[0,T]_{\mathbb{T}}}\left\{\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!}\right.\right.$$

$$\left.+\left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\left|f\left(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s\right)\right.$$

$$\left.-f\left(t,y(t),y^{\Delta}(t),\cdots,y^{\Delta^{n-1}}(t),\int_0^t g(t,s,y(s),\cdots,y^{\Delta^{n-1}}(s))\Delta s\right)\right|\Delta s$$

$$-\int_0^t\frac{(t-\sigma(s))^{n-1}}{(n-1)!}\left|f\left(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s)\right)\Delta s\right.$$

$$\left.-f\left(t,y(t),y^{\Delta}(t),\cdots,y^{\Delta^{n-1}}(t),\int_0^t g(t,s,y(s),\cdots,y^{\Delta^{n-1}}(s))\Delta s\right)\right|\Delta s$$

$$-\sum_{0<t_k<t}|I_k(x(t_k))-I_k(y(t_k))|\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!}+\left(\frac{t-t_k}{2}\right)^{n-1}\right]$$

$$\left.+\sum_{0<t_k<t}|I_k(x(t_k))-I_k(y(t_k))|\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$

$$\leq\left[L||x-y||_{PC^{n-1}}+LM||x-y||_{PC^{n-1}}\right]\sup_{t\in[0,T]_{\mathbb{T}}}\left\{\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!}+\left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\Delta s\right.$$

$$\left.-\int_0^t\frac{(t-\sigma(s))^{n-1}}{(n-1)!}\Delta s\right\}+Q||x-y||\sup_{t\in[0,T]_{\mathbb{T}}}\left\{-\sum_{0<t_k<t}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!}+\left(\frac{t-t_k}{2}\right)^{n-1}\right]\right.$$

$$\left.+\sum_{0<t_k<t}\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$

Taking maximum over the interval $[0,T]_{\mathbb{T}}$. We get

$$||(Fx)-(Fy)||\leq L\gamma||x-y||_{PC_{n-1}}+LM\gamma||x-y||_{PC_{n-1}}+Q\beta||x-y||_{PC_{n-1}}.$$

Where γ and β is given by (10). By the assumption: $L\gamma+LM\gamma+Q\beta<1$, we deduce that F is a contraction. Hence, by the contraction mapping principle, (1) has a unique solution.

3.2 Theorem

Assume that (H_2) and (H_4) holds. Then the problem (1) has atleast one solution on $[0,T]_{\mathbb{T}}$.

Proof: We show that the operator F denoted by (9) satisfies the hypothesis of Schauder's fixed point theorem. This will be done in several steps.

Step 1: F is continuous.

Let $\{x_k\}$ be a sequence such that $x_k\to x$ in $C([0,T]_{\mathbb{T}},\mathbb{R}^n)$. Then, for each $t\in[0,T]_{\mathbb{T}}$, we have

$|(F)(x_k)(t)-(Fx)(t)|$

$$\leq\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!}\right.$$

$$\left.+\left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\left|f\left(t,x_k(t),x_k^{\Delta}(t),\cdots,x_k^{\Delta^{n-1}}(t),\int_0^t g(t,s,x_k(s),\cdots,x_k^{\Delta^{n-1}}(s))\Delta s\right)\right.$$

$$-f\left(t, x(t), x^{\Delta}(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t, s, x(s), \cdots, x^{\Delta^{n-1}}(s))\Delta s\right)\Bigg| \Delta s$$

$$-\int_0^t \frac{\left(t-\sigma(s)\right)^{n-1}}{(n-1)!}\left|f\left(t, x_k(t), x_k^{\Delta}(t), \cdots, x_k^{\Delta^{n-1}}(t), \int_0^t g(t, s, x_k(s), x_k^{\Delta}(s), \cdots, x_k^{\Delta^{n-1}}(s)\right)\Delta s\right.$$

$$\left.- f\left(t, x(t), x^{\Delta}(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t, s, x(s), \cdots, x^{\Delta^{n-1}}(s))\Delta s\right)\right| \Delta s$$

$$-\sum_{0<t_k<t} |I_k(x_k(t_k)) - I_k(x(t_k))| \left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right]$$

$$\left. + \sum_{0<t_k<t} |I_k(x_k(t_k)) - I_k(x(t_k))| \frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$

$$\leq ||f(.,x_k(.))$$

$$- f(.,x(.))|| \sup_{t\in[0,T]_{\mathbb{T}}} \left\{\int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\frac{\left(t-\sigma(s)\right)^{n-2}}{(n-2)!} + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\Delta s - \int_0^t \frac{\left(t-\sigma(s)\right)^{n-1}}{(n-1)!}\Delta s\right\}$$

$$+||I_k(x_k(.)) - I_k(x(.))|| \sup_{t\in[0,T]_{\mathbb{T}}} \left\{-\sum_{0<t_k<t}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t}\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$

$$\leq \gamma \left|\left|f(.,x_k(.)) - f(.,x(.))\right|\right| + \beta \left|\left|I_k(x_k(.)) - I_k(x(.))\right|\right| \to 0 \text{ as } k \to \infty.$$

Therefore F is continuous.

Step 2: F maps bounded sets into bounded sets in $C([0,T]_{\mathbb{T}}, \mathbb{R}^n)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant P such that for each $x \in B_{\eta}^* = \left\{x \in C([0,T]_{\mathbb{T}}, \mathbb{R}^n): ||x||_{PC^{n-1}} \leq \eta^*\right\}$,

we have $||Fx|| \leq P$. For each $t \in [0,T]_{\mathbb{T}}$ by the condition (H_4), we have

$$|(Fx)(t)| = \int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\frac{\left(t-\sigma(s)\right)^{n-2}}{(n-2)!} + \left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]\left|f\left(t, x(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t, s, x(s), \cdots, x^{\Delta^{n-1}}(s))\Delta s\right)\right| \Delta s$$

$$-\int_0^t \frac{\left(t-\sigma(s)\right)^{n-1}}{(n-1)!}\left|f(t, x(t), x^{\Delta}(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t, s, x(s), x^{\Delta}(s), \cdots, x^{\Delta^{n-1}}(s))\Delta s)\right| \Delta s$$

$$-\sum_{0<t_k<t} |I_k(x(t_k))| \left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!} + \left(\frac{t-t_k}{2}\right)^{n-1}\right] + \sum_{0<t_k<t} |I_k(x(t_k))| \frac{(t-t_k)^{n-1}}{(n-1)!}$$

Taking the norm for $t \in [0,T]_{\mathbb{T}}$, the above inequality yields $||F(x)|| \leq (\gamma + \beta)$, where γ and β is given by (10).

Step 3: F maps bounded sets into equicontinuous sets of $C([0,T]_{\mathbb{T}}, \mathbb{R}^n)$.

Let $t_1, t_2 \in (t_k, t_{k+1})$, $t_1 < t_2$ and B_{η^*} be a bounded set in $C([0,T]_{\mathbb{T}}, \mathbb{R}^n)$ as in Step 2. Then for $\in B_{\eta^*}$, we have

$$|F(x)(t_2) - F(x)(t_1)|$$

$$\leq \left|\int_0^{\sigma(T)} \left[\left(\frac{\sigma(T)}{4}\right)\frac{\big(t_2-\sigma(s)\big)^{n-2} - \big(t_1-\sigma(s)\big)^{n-2}}{(n-2)!} + \left(\frac{t_2-\sigma(s)}{2}\right)^{n-1}\right.\right.$$

$$\left.\left. - \left(\frac{t_1-\sigma(s)}{2}\right)^{n-1}\right] f\left(t, x(t), \cdots, x^{\Delta^{n-1}}(t), \int_0^t g(t,s,x(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s\right)\Delta s\right|$$

$$-\left|\int_0^{t_1} \frac{\big(t_2-\sigma(s)\big)^{n-1} - \big(t_1-\sigma(s)\big)^{n-1}}{(n-1)!} f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t), \int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s)\Delta s\right.$$

$$\left.+\int_{t_1}^{t_2} \frac{\big(t_2-\sigma(s)\big)^{n-1}}{(n-1)!} f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t), \int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s)\Delta s\right|$$

$$-\left|\sum_{0<t_k<t} I_k(x(t_k))\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t_2-t_k)^{n-2}-(t_1-t_k)^{n-2}}{(n-2)!} + \left(\frac{t_2-t_k}{2}\right)^{n-1} - \left(\frac{t_1-t_k}{2}\right)^{n-1}\right]\right|$$

$$+\left|\sum_{0<t_k<t_1} I_k(x(t_k))\frac{(t_2-t_k)^{n-1}-(t_1-t_k)^{n-1}}{(n-1)!} + \sum_{t_1<t_k<t_2} I_k(x(t_k))\frac{(t_2-t_k)^{n-1}}{(n-1)!}\right|$$

$$\leq N\left|\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{\big(t_2-\sigma(s)\big)^{n-2}-\big(t_1-\sigma(s)\big)^{n-2}}{(n-2)!} + \left(\frac{t_2-\sigma(s)}{2}\right)^{n-1} - \left(\frac{t_1-\sigma(s)}{2}\right)^{n-1}\right]\Delta s\right|$$

$$- N\left|\int_0^{t_1}\frac{\big(t_2-\sigma(s)\big)^{n-1}-\big(t_1-\sigma(s)\big)^{n-1}}{(n-1)!}\Delta s + \int_{t_1}^{t_2}\frac{\big(t_2-\sigma(s)\big)^{n-1}}{(n-1)!}\Delta s\right|$$

$$-R\left|\sum_{0<t_k<t}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t_2-t_k)^{n-2}-(t_1-t_k)^{n-2}}{(n-2)!} + \left(\frac{t_2-t_k}{2}\right)^{n-1} - \left(\frac{t_1-t_k}{2}\right)^{n-1}\right]\right|$$

$$+R\left|\sum_{0<t_k<t_1}\frac{(t_2-t_k)^{n-1}-(t_1-t_k)^{n-1}}{(n-1)!} + \sum_{t_1<t_k<t_2} I_k(x(t_k))\frac{(t_2-t_k)^{n-1}}{(n-1)!}\right|$$

$$\leq N\left|\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{\big(t_2-\sigma(s)\big)^{n-2}-\big(t_1-\sigma(s)\big)^{n-2}}{(n-2)!} + \left(\frac{t_2-\sigma(s)}{2}\right)^{n-1} - \left(\frac{t_1-\sigma(s)}{2}\right)^{n-1}\right]\Delta s\right|$$

$$-\frac{N}{n!}[2(t_2-t_1)^n + |t_2^n - t_1^n|] - R\left|\sum_{0<t_k<t}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t_2-t_k)^{n-2}-(t_1-t_k)^{n-2}}{(n-2)!} + \left(\frac{t_2-t_k}{2}\right)^{n-1}\right.\right.$$

$$\left.\left.-\left(\frac{t_1-t_k}{2}\right)^{n-1}\right]\right| + \frac{R}{(n-1)!}\sum_{0<t_k<t_1}[2(t_2-t_1)^{n-1} + |t_2^{n-1} - t_1^{n-1}|]$$

Clearly the right hand side of the above inequality tends to zero independent of x as $(t_2 - t_1) \to 0$. In view of the above three steps, the Arzela-Ascoli theorem applies and consequently the operator $F: C([0,T]_{\mathbb{T}}, \mathbb{R}^n) \to C([0,T]_{\mathbb{T}}, \mathbb{R}^n)$ is continuous and completely continuous.

Step 4: To obtain priori bound of the set $\epsilon = \{x \in PC([0,T]_{\mathbb{T}}, \mathbb{R}^n): x = \lambda F(x), \lambda \in (0,1)\}$ is bounded. Let $x \in \epsilon$, then $x = \lambda F(x)$ for some $0 < \lambda < 1$. Then for each $t \in [0,T]_{\mathbb{T}}$, we have

$$x(t)=\lambda\left\{\int_0^{\sigma(T)}\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-\sigma(s))^{n-2}}{(n-2)!}+\left(\frac{t-\sigma(s)}{2}\right)^{n-1}\right]f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s)\Delta s\right.$$

$$-\int_0^t\frac{(t-\sigma(s))^{n-1}}{(n-1)!}f(t,x(t),x^{\Delta}(t),\cdots,x^{\Delta^{n-1}}(t),\int_0^t g(t,s,x(s),x^{\Delta}(s),\cdots,x^{\Delta^{n-1}}(s))\Delta s)\Delta s$$

$$\left.-\sum_{0<t_k<t}I_k(x(t_k))\left[\left(\frac{\sigma(T)}{4}\right)\frac{(t-t_k)^{n-2}}{(n-2)!}+\left(\frac{t-t_k}{2}\right)^{n-1}\right]+\sum_{0<t_k<t}I_k(x(t_k))\frac{(t-t_k)^{n-1}}{(n-1)!}\right\}$$

Using the condition (H_4), it is easy to shows that $||F(x)|| \leq N\gamma + R\beta$ This shows that ϵ is bounded. Thus, it follows by Schauder's fixed point theorem the operator F has a fixed point, which is a solution of a problem (1).

4. EXAMPLES

4.1 Example

Consider the problem

$$x^{\Delta^2}(t)=\frac{1}{4}t^3(x(t)-2)+\frac{5}{18}\left(\int_0^t t^2s^4x(s)\Delta s\right)^2, t\in[0,1]_{\mathbb{T}}, t\neq\frac{1}{2},$$

$$x(0)=-x(\sigma(1)), x^{\Delta}(0)=-x^{\Delta}(\sigma(1)),$$

$$x\left(\left(\frac{1}{2}\right)^+\right)=x\left(\frac{1}{2}\right)+\frac{1}{5}x\left(\frac{1}{2}\right), x^{\Delta}\left(\left(\frac{1}{2}\right)^+\right)=x^{\Delta}\left(\frac{1}{2}\right)+\frac{1}{5}x^{\Delta}\left(\frac{1}{2}\right),$$

Solution: Here $n=2, f(t,x,y)=\frac{1}{4}t^3(x-2)+\frac{5}{18}y^2$. Using the given data, we find $L=\frac{1}{4}, M=\frac{1}{3}$ and $Q=\frac{1}{5}$ as $|f(t,x_2,y_2)-f(t,x_1,y_1)|\leq\frac{1}{4}||x_2-x_1||+\frac{1}{3}||y_2-y_1||, |I_k(x_2(t_k))-I_k(x_1(t_k))|\leq\frac{1}{5}||x_2(t_k)-x_1(t_k)||, \gamma=0.25$ and $\beta=0.1$.

Obviously $L\gamma + LM\gamma + Q\beta \approx 0{:}103333 < 1$. Thus all the conditions of Theorem (3.1) are satisfied. Hence, by the conclusion of Theorem (3.1), problem (11) has a unique solution on $[0,1]_{\mathbb{T}}$.

4.2 Example

Consider the problem

$$x^{\Delta^4}(t)=\frac{e^{-x^2(t)}+2\sin(1+3x(t))+\cos(3+5x^3(t))+3x^4}{1+x^2(t)}, t\in[0,1]_{\mathbb{T}},$$

$$x(0)=-x(\sigma(1)), x^{\Delta}(0)=-x^{\Delta}(\sigma(1)), x^{\Delta^2}(0)=-x^{\Delta^2}(\sigma(1)), x^{\Delta^3}(0)=-x^{\Delta^3}(\sigma(1))$$

$$x\left(\left(\frac{1}{2}\right)^+\right)=x\left(\frac{1}{2}\right)+\frac{1}{5}x\left(\frac{1}{2}\right), x^{\Delta}\left(\left(\frac{1}{2}\right)^+\right)=x^{\Delta}\left(\frac{1}{2}\right)+\frac{1}{5}x^{\Delta}\left(\frac{1}{2}\right),$$

$$x^{\Delta^2}\left(\left(\frac{1}{2}\right)^+\right)=x^{\Delta^2}\left(\frac{1}{2}\right)+\frac{1}{5}x^{\Delta^2}\left(\frac{1}{2}\right), x^{\Delta^3}\left(\left(\frac{1}{2}\right)^+\right)=x^{\Delta^3}\left(\frac{1}{2}\right)+\frac{1}{5}x^{\Delta^3}\left(\frac{1}{2}\right)$$

Solution: Here $f(t,x(t))=\frac{e^{-x^2(t)}+2\sin(1+3x(t))+\cos(3+5x^3(t))+3x^4}{1+x^2(t)}$. Clearly $f(t,x(t))$ is continuous and $|f(t,x(t))|\leq N$ with $N=7$ for each $t\in[0,1]_{\mathbb{T}}$ and all $x\in\mathbb{R}$. Thus the conclusion of Theorem 3.2 applies and the problem (12) has a solution on $t\in[0,1]_{\mathbb{T}}$.

5. CONCLUSION

In this study, we investigated and built a system of higher order impulsive integro-differential equations on time scales subject to antiperiodic boundary value problems on time scales. The integral term has been included to the system of equations to make the equation more clear. We arrived at the distinctive solution by applying the green's function technique and deduced the existence of solutions by utilizing the Contraction mapping principle and Leray Schauder's fixed point theorem. In addition, an example is provided to demonstrate the effectiveness of the result.

REFERENCES

[1] Ahmad, B. Ntouyas, S, K. and Alsulami, H, H. 2013. Existence results for nth order multi point integral boundary value problems of differential inclusions. Electronic Journal of Differential equations, 203: 13 pages.

[2] Arunkumar, D. Sivabalan, M. Sathiyanathan, K. 2019. Existence of Impulsive Integro differential equations with integral boundary conditions. American international Journal of Research in science, Technology, Engineering and Mathematics: 5-13.

[3] Eloe, PW. and Ahmad, B. 2005. Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions. Applied Mathematical Letters, 18: 521-527.

[4] Karaca, IY. and Fen, FT. 2015. Positive solutions of nth order boundary value problems with integral boundary conditions. Mathematical modelleing and analysis, 20: 188-204.

[5] Krishnaveni, V. Sathiyanathan, K. Arunkumar, D. 2016. Impulsive Integro differential equation with antiperiodic boundary conditions on time scales. International Journal of Control theory and Applications, 9: 1136-1150.

[6] Lakshmikantham, V. Bainov, D. Simeonov, P. 1989. Theory of Impulsive Differential Equations, World Scientific, Singapore.

[7] Li, Y. Zhang, H. 2014. Positive solutions for a nonlinear higher order differential system with coupled integral boundary conditions. Journal of applied Mathematics: 7 pp.

[8] Li, Y. Li, F. 2008. Sign-changing solutions to second-order integral boundary value problems. Nonlinear Analysis, 69: 1179-1187.

[9] Liu, X. Guo, D. 1997. Periodic boundary value problems for a class of second order impulsive integro-differential equations in Banach space. Journal of Mathematical Analysis and Applications, 216: 284-302.

[10] Thaiprayoon, C. Samana, D. Tariboon, J. 2012. Periodic boundary value problems for second order impulsive integro differential equations with integral jump conditions. Boundary value problems, 122.

[11] Tian, Y. Ji, D. Ge, W. 2009. Existence and nonexistence results of impulsive first order problem with integral boundary condition. Nonlinear Analysis, 71: 1250-1262.

[12] Wang, W. Lingling, Z. Zhandong, L. 2006. Initial value problems for nonlinear impulsive integro-differential equations in Banach space. Journal of Mathematical Analysis and Applications, 3720: 510-527.

[13] Webb, JRL. 2009. Positive solutions of some higher order nonlocal boundary value problems. Electronic Journal of Qualitative Theory of Differential Equations, 29: 15pp.

[14] Zhang, X. Yang, X. Ge, W. 2009. Positive solutions of nth-order impulsive boundary value problems with integral boundary conditions in Banach spaces. Nonlinear analysis,71: 5930-5945.