Journal of Neonatal Surgery
ISSN(Online): 2226-0439
Vol. 14, Issue 29s (2025)
https://www.jneonatalsurg.com

OPEN ACCESS

Solving Fractional Oscillator Equations Via The Sumudu Transform Method

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Cite this paper as: Obaidullah Jaihon, (2025) Solving Fractional Oscillator Equations Via The Sumudu Transform Method. *Journal of Neonatal Surgery*, 14 (29s), 739-746.

ABSTRACT

The equation of motion for a driven fractional oscillator is formulated by replacing the classical second-order time derivative with a Caputo fractional derivative of order $1 < \alpha \leq 2$. In this study, the Sumudu transform method is employed to obtain an analytical solution of the fractional differential equation. By utilizing the integral properties of the Sumudu transform, which are adapted for Caputo fractional derivatives, the system's response is explicitly derived in the time domain. The dynamic characteristics and phase plane trajectories of the fractional oscillator are analyzed for various values of α. The results demonstrate that the Sumudu transform provides an effective and accurate framework for analyzing the complex behavior of fractional oscillatory systems.

Keywords: *Fractional differential equation, Caputo fractional derivative, Equation of motion, Sumudu transform method.*

1. INTRODUCTION

Recently, fractional calculus has emerged as a sophisticated and versatile branch of mathematics, garnering significant interest across various scientific disciplines. This field extends the traditional concepts of differentiation and integration to non-integer and arbitrary orders, enabling more flexible and accurate modeling of complex phenomena. Its wide-ranging applications in physics, engineering, and natural sciences—particularly in the analysis of intricate and fractal-like systems—have elevated its importance considerably[1]. Numerous studies have thoroughly explored the role of fractional calculus in continuum mechanics and statistical mechanics, highlighting its theoretical and practical relevance.

A key model in this context is the driven fractional oscillator, in which the traditional second-order time derivative of the simple harmonic oscillator equation is substituted by a fractional derivative of order α, where $1 < \alpha \leq 2$. These fractional derivatives are defined using the Caputo approach, which allows the system to display a variety of dynamic behaviors depending on the parameter αα. Notably, when αα equals 2, the model simplifies to the classical harmonic oscillator([2], [3],[4], [5]).

A range of analytical and numerical methods, including the Laplace transform and Adomian decomposition, have been developed to address fractional differential equations. However, the Sumudu transform has recently attracted interest as an effective and innovative approach, notable for maintaining physical units, simplifying calculations, and lowering computational effort. These features make it a compelling alternative to the Laplace transform for tackling fractional differential equations ([6], [7]). The Sumudu transform has demonstrated effectiveness in solving both linear and nonlinear fractional differential equations, as well as fractional partial differential equations, widely applied with success in engineering and applied science fields([8],[9]).

Recent investigations reveal that employing the Sumudu transform not only enhances computational efficiency but also yields accurate analytical and approximate solutions for fractional equations [7].Furthermore, integrating the Sumudu transform with modern iterative methods facilitates semi-analytical solutions to nonlinear fractional differential equations, expanding its utility in complex problem-solving [10].

The practical applications of fractional calculus and the Sumudu transform extend well beyond pure mathematics. These methods have found use in various domains including linear and nonlinear control, fractional filter development, modeling of batteries and supercapacitors, seismic event prediction, nuclear interaction analysis, and modeling of epidemics..For instance, in particle physics, the interplay between fractional oscillators and q-deformed models underscores the significant role of fractional calculus in elucidating nuclear interactions [11].Additionally, due to the fractal characteristics inherent in geological formations, earthquake forecasting has become feasible through fractional calculus methodologies.

In the present study, the equation governing the driven fractional oscillator is transformed into the Sumudu domain, enabling the derivation of an analytical solution expressed in terms of system parameters. Subsequently, applying the inverse Sumudu transform retrieves the time-domain response of the system. This approach facilitates precise and efficient analysis of fractional oscillators and holds considerable promise for modeling complex systems in engineering and physics.

This paper is structured in the following manner. Section 1 offers an overview of fractional calculus theory. Section 2 outlines the fundamental definitions and preliminaries. Section 3 employs the Sumudu transform technique to obtain analytical solutions for the driven fractional oscillator.

2. DEFINITIONS AND PRELIMINARIES

Definition 2. 1([[12],[13]]). For a causal function $f(t)$, the fractional derivative is given by.

$$\frac{d^\beta}{dt^\beta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau\,, & if \;\; n-1<\beta<n, \\ f^{(n)}(x) \qquad , & if \;\; \beta = n \in N, \end{cases}$$

where $\Gamma(\cdot)$ denotes the Euler gamma function, defined by

$$\Gamma(\mathrm{x}) = \int_0^\infty t^{x-1} e^{-t} dt \quad (\mathbb{R}(x) > 0).$$

Definition 2. 2([14],[15]).The Sumudu transform operates on the set of functions

$$A = \{\varphi(\tau) \;/\exists M, \omega_1, \omega_2 > 0, |\varphi(\tau)| < M e^{\frac{|\tau|}{\omega_j}}, if \;\; \tau \in (-1)^j \times [0,\infty)$$

by the following formula

$$S[\varphi(\tau)](\omega) = \int_0^\infty e^{-t}\, \varphi(\omega\tau) d\tau, \qquad \omega \in (\omega_1, \omega_2).$$

Definition 2. 3 [16]. If $p(t)$ and $q(t)$ are exponential order and $P(u)$ and $Q(u)$ are the Sumudu transform of $p(t)$ and $q(t)$ respectively, then

$$S[p(t) * q(t)](u) = S\left[\int_0^t q(\tau) p(t-\tau)\, d\tau\right](u) = uQ(u)P(u)$$

Definition 2. 4[16]. If $g^{(n)}(t)$ is of exponential order then the Sumudu transform of the n'th derivative of g is given by

$$S[g^n(t)] = \frac{1}{u^n}[G(u) - \sum_{k=0}^{m-1} u^k f^k(0)]$$

Definition 2. 5 ([17],[18]). The Mittag-Leffler functions $E_\mu(.)$ and $E_{\mu,v}(.)$ Are respectively defined by the following series pair

$$E_\mu(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\mu+1)} \;\; , z \in \mathbb{C}, \mathbb{R}\,(\mu > 0)$$

and

$$E_{\mu,v}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\mu+v)} \;\; , z, v \in \mathbb{C}, \mathbb{R}\,(\mu > 0).$$

Definition2.5[19]. The Beta function, also termed Euler's first kind integral

$$\beta(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds, \quad Re(x) > 0\,, \quad Re(y) > 0.$$

Here, the Sumudu transform of the Caputo fractional derivative is derived

Lemma2.1([20],[21]).

$$S\left[D_t^\beta f(t)\right] = u^{-\beta}\left[F(u) - \sum_{k=0}^{n-1} u^k f^k(0)\right]$$

Proof: Employing Definitions 3 and 4, we find

$$S[D_t^{\beta} f(t)] = S[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{m-\beta-1} f^{(n)}(\tau) d\tau,$$

$$= \frac{u}{\Gamma(n-\alpha)} S[f^{(n)}(t) S[t^{n-\beta-1}]$$

$$= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{1}{u^{n-1}} \left[F(u) - \sum_{k=0}^{m-1} u^k f^k(0) \right] \cdot \frac{u^{n-1} \cdot u^{-\beta}}{\Gamma(n-\alpha)}$$

$$= u^{-\beta} \left[F(u) - \sum_{k=0}^{n-1} u^k f^k(0) \right] .$$

Lemma2.2[17]. Considering any element of the complex plane C for any $R\{\alpha\} > 0, R\{\beta\} > 0$ and $B \in \mathbb{C}^{n \times n}$.

$$S[t^{\beta-1} E_{\alpha,\beta}(Bt^{\alpha})] = \frac{u^{\beta-1}}{1 - Bu^{\alpha}}$$

holds for $\mathbb{R}[u] > 1 \ / \ ||B||^{1/\alpha}$.

Proof. For $\mathbb{R}[u] > 1 \ / \ ||B||^{1/\alpha}$ we have

$$S[t^{\beta-1} E_{\alpha,\beta}(Bt^{\alpha})] = \int_0^{\infty} e^{-t} (ut)^{\beta-1} \sum_{k=0}^{\infty} \frac{(B(ut)^{\alpha})^k}{\Gamma(\alpha k + \beta)} dt$$

$$= u^{\beta-1} \sum_{k=0}^{\infty} \frac{B^k u^{\alpha k}}{\Gamma(\alpha k + \beta)} \int_0^{\infty} e^{-t} t^{\alpha k + \beta - 1} dt$$

$$= u^{\beta-1} \sum_{k=0}^{\infty} \frac{B^k u^{\alpha k}}{\Gamma(\alpha k + \beta)} \Gamma(\alpha k + \beta)$$

$$= \frac{u^{\beta-1}}{1 - Bu^{\alpha}} .$$

Now we consider for real $\beta > 0$ (later only for $1 < \beta \leq 2$), the fractional differential equation

$$\frac{d^{\alpha} y}{dt^{\alpha}} + \omega^{\alpha} y(t) = f(t), \qquad n-1 < \beta \leq n \tag{2.1}$$

subject to the initial conditions

$y^{(k)}(0) = c_k$, $\quad k = 0,1, \dots, n-1.$ (2.2)

In equation (2.1), ω is an arbitrary constant, and $f(t)$ is a continuous function. The integer mm is uniquely determined by the inequality $n-1 < \alpha \leq n$, which specifies the number of initial conditions $y^k(0) = c_k \, for \, k = 0,1, \dots, n-1$. Depending on the value of α, this equation is classified as follows: for $1 < \alpha \leq 2$, it is known as the fractional oscillation equation, for $0 < \alpha < 1$, it corresponds to the fractional relaxation equation; and for $2 < \alpha \leq 3$, it is referred to as the fractional growing oscillation equation.

The fractional derivative in this equation is defined using the Caputo formulation, which ensures the existence of a unique solution under the given initial conditions. Unlike the Riemann–Liouville definition—which requires initial conditions expressed via fractional integrals and derivatives that often lack clear physical interpretation—the Caputo derivative employs initial conditions in terms of integer-order derivatives. This feature makes the Caputo definition generally more suitable and widely preferred for modeling physical systems([22],[23]). For a comprehensive discussion on the existence and uniqueness of solutions to fractional differential equations, see .

3. EVALUATION OF THE ANALYTICAL METHOD

Our focus in this section is the development of an efficient algorithm designed for solving a general system of driven fractional oscillators

$$\frac{d^{\alpha} x}{dt^{\alpha}} + \omega^{\alpha} x(t) = f(t), \quad 1 < \alpha \leq 2 \tag{3.1}$$

Assuming initial values

$$x(0) = a \quad x'(0) = b, \tag{3.2}$$

where ω is the natural frequency and $f(t)$ is the forcing function.

From Lemma 2.1 we have

$$S\left[\frac{d^\alpha x}{dt^\alpha}\right] + \omega^\alpha S[x(t)] = S[f(t)] \tag{3.3}$$

$$u^{-\alpha}[X(u) - a - bu] + \omega^\alpha X(u) = F(u)$$

$$X(u)\left[\frac{1}{u^\alpha} + \omega^\alpha\right] = F(u) + au^\alpha + bu^{\alpha+1}$$

$$X(u) = \frac{F(u) + au^\alpha + bu^{\alpha+1}}{\frac{1}{u^\alpha} + \omega^\alpha}$$

$$= \frac{a + bu + F(u)u^\alpha}{1 + \omega^\alpha u^\alpha}$$

Or

$$X(u) = \frac{a}{1 + \omega^\alpha u^\alpha} + + \frac{bu}{1 + \omega^\alpha u^\alpha} + \frac{F(u)u^\alpha}{1 + \omega^\alpha u^\alpha} \tag{3.4}$$

From Lemma 2.2 and Definition 2.3, it follows that

$$x(t) = aE_{\alpha,1}(-\omega^\alpha t^\alpha) + btE_{\alpha,2}(-\omega^\alpha t^\alpha) + \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(-\omega^\alpha (t-u)^\alpha) f(u)du. \tag{3.5}$$

Example 3.1 Let's study the following fractional differential equation

$$\frac{d^\alpha x}{dt^\alpha} + \omega^\alpha x(t) = 0 \ , \qquad 1 < \alpha \le 2 \tag{3.6}$$

using initial values

$$x(0) = 1 \ \ , \qquad x'(0) = 0. \tag{3.7}$$

This equation describes a simple harmonic fractional oscillator where the forcing function in this case is $f(t) = 0$.

From Theorem 2.2 we get

$$S[\frac{d^\alpha x}{dt^\alpha}] + \omega^\alpha S[x(t)] = 0$$

$$u^{-\alpha}[X(u) - a] + \omega^\alpha X(u) = 0$$

or

$$X(u) = \frac{a}{1 + \omega^\alpha u^\alpha}$$

From Definiton 2.5 and Lemma 2.2 we have

$$x(t) = E_{\alpha,1}(-\omega^\alpha t^\alpha).$$

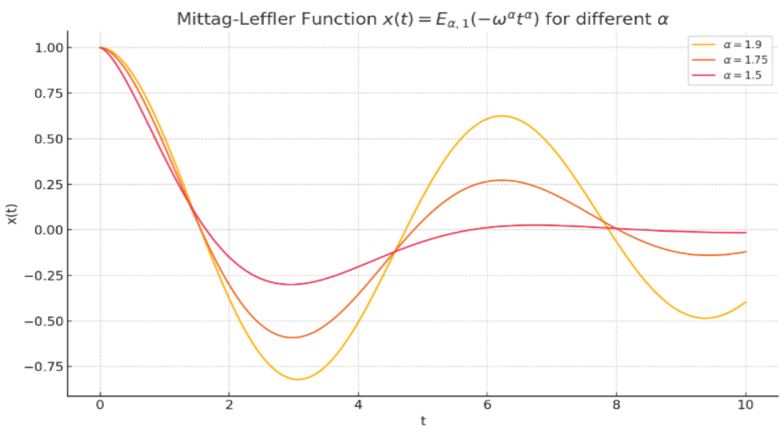

Figure 1. Response function ofequation (3.6) for different values of α

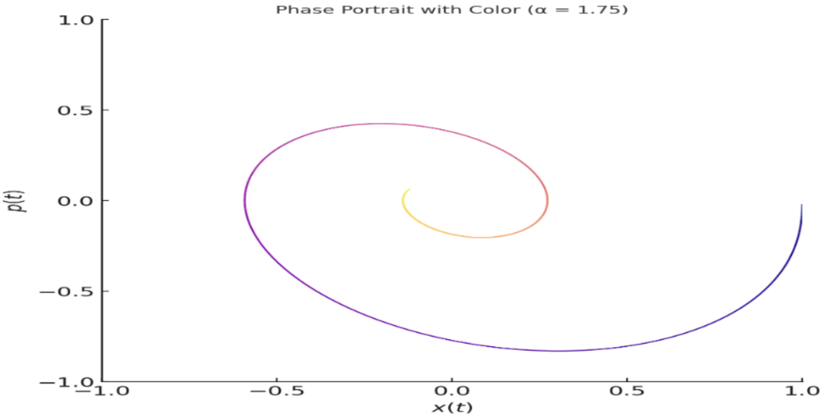

Figure 2. Response function ofequation (3.6) of α =1.75

Figure 1 depicts the system's evolution for various values of αα, highlighting how the displacement of the fractional oscillator changes over time and how this behavior depends on the parameter αα. The results show that the driven fractional oscillator exhibits characteristics similar to a damped harmonic oscillator: the motion remains oscillatory, but the total energy decreases over time. Additionally, the phase plane trajectory no longer forms a closed loop; instead, it traces a logarithmic spiral, reflecting the intrinsic damping and memory effects typical of fractional dynamics. This aligns with the understanding that fractional oscillators generalize classical harmonic oscillators by incorporating algebraic decay and nonlocal memory effects, resulting in a gradual energy loss and altered phase-space behavior.

Figures 2 and 3 depict the phase plane trajectories for $\alpha = 1.7$, *where a comparable behavior to that of the damped oscillator is also observed.*

Example 3.2 *Now, we examine the fractional differential system describing oscillations in the following form:*

$$\frac{d^\alpha y}{dt^\alpha} + \omega^\alpha x(t) = g(t)\,, \qquad 1 < \alpha \le 2 \tag{3.8}$$

Assuming initial conditions

$$y(0) = a \quad , \qquad y'(0) = 0, \tag{3.9}$$

in this case the forcing term is represented by the step function

$$g(t) = \begin{cases} B, & t > 0, \\ 0, & t < 0. \end{cases}$$

From Theorem 2.2 we get

$$S[\frac{d^\alpha y}{dt^\alpha}] + \omega^\alpha S[y(t)] = S[g(t)]$$

$$u^{-\alpha}[Y(u) - a] + \omega^\alpha Y(u) = B$$

$$Y(u)\left[\frac{1}{u^\alpha} + \omega^\alpha\right] = B + au^{-\alpha}$$

or

$$X(u) = \frac{a}{1 + \omega^\alpha u^\alpha} + B\frac{u^\alpha}{1 + \omega^\alpha u^\alpha}$$

From Lemma 2.2 and Definition 2.3 we have

$$x(t) = aE_{\alpha,1}(-\omega^\alpha\, t^\alpha) + B\int_0^t (t-u)^{\alpha-1}\, E_{\alpha,\alpha}((-\omega^\alpha(t-u)^\alpha)du$$

or

$$y(t) = \sum_{k=0}^{\infty}(-\omega)^{\alpha k}\left[a\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)} + B\sum_{k=0}^{\infty}\frac{t^{\alpha(k+1)}}{\Gamma(\alpha k+\alpha+1)}\right]. \tag{3.10}$$

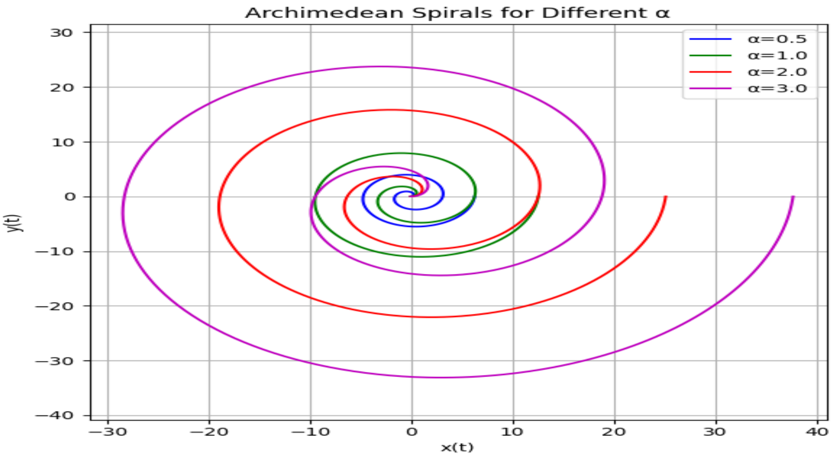

Figure 3.Response function ofequation (3.8) for different value of α

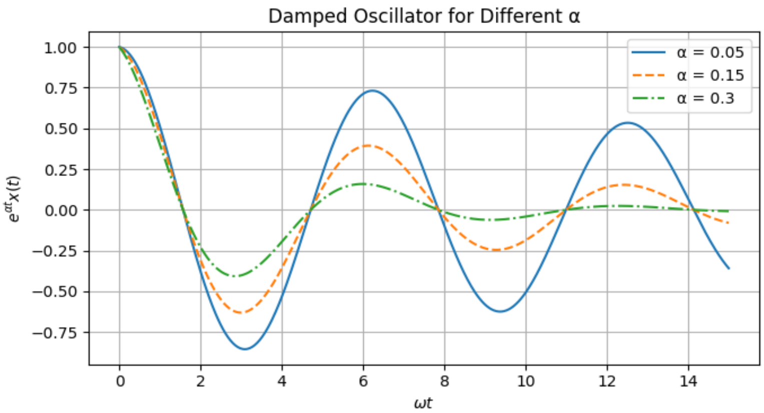

Figure 4.Response function ofequation (3.8) for different value of α

Example 3.3 *In this example, we select the forcing function as the sinusoidal function* $f(t) = sin(\lambda t)$ *and express equation (3.1) accordingly.*

$$\frac{d^\alpha y}{dt^\alpha} + \lambda^\alpha y(t) = sin(\omega t) \quad , \qquad 1 < \alpha \leq 2 \tag{3.11}$$

using initial conditions

$$y(0) = 0 \quad , \qquad y'(0) = 0. \tag{3.12}$$

Above mentioned, from Theorem 2.2 we get

$$S[\frac{d^\alpha y}{dt^\alpha}] + \lambda^\alpha S[y(t)] = S[sin(\lambda t)]$$

$$u^{-\alpha} Y(u) + \lambda^\alpha Y(u) = \frac{\lambda u}{1 + \lambda^2 u^2}$$

$$Y(u)\left[\frac{1}{u^\alpha} + \lambda^\alpha\right] = \frac{\lambda u}{1 + \lambda^2 u^2}$$

or

$$Y(u) = \frac{\lambda u}{(1 + \lambda^\alpha u^\alpha)(1 + \omega^2 u^2)}.$$

By using Lemma 2.2 and Definition 2.3 we have

$$y(t) = \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(-\lambda^\alpha (t-u)^\alpha) \sin(\lambda u)\, du. \quad (3.3)$$

Hence we get

$$y(t) = \int_0^t (t-u)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k} (t-u)^{\alpha k}}{\Gamma(\alpha k+\beta)} \sin(\lambda u)\, du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k}}{\Gamma(\alpha k+\beta)} \int_0^t (t-u)^{\alpha k+\alpha-1} \sin(\lambda u)\, du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k}}{\Gamma(\alpha k+\beta)} \sum_{k=0}^{\infty} \frac{(-1)^j \lambda^{2j+1}}{\Gamma(2j+2)} \int_0^t (t-u)^{\alpha k+\alpha-1} \tau^{2j+1} du,$$

from Definition 2.5 we get

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k}}{\Gamma(\alpha k+\beta)} \sum_{k=0}^{\infty} \frac{(-1)^j \omega^{2j+1}}{\Gamma(2j+2)} \cdot \frac{\Gamma(\alpha k+\alpha)\Gamma(2j+2)}{\Gamma(\alpha(k+1)+2j+2))} t^{\alpha(k+1)+2j+1}.$$

$$= \sum_{k=0}^{\infty} (-1)^k \omega^{\alpha k} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(\alpha(k+1)+2j+2))} (\lambda t)^{2j+1+\alpha(k+1)} \quad (3.4)$$

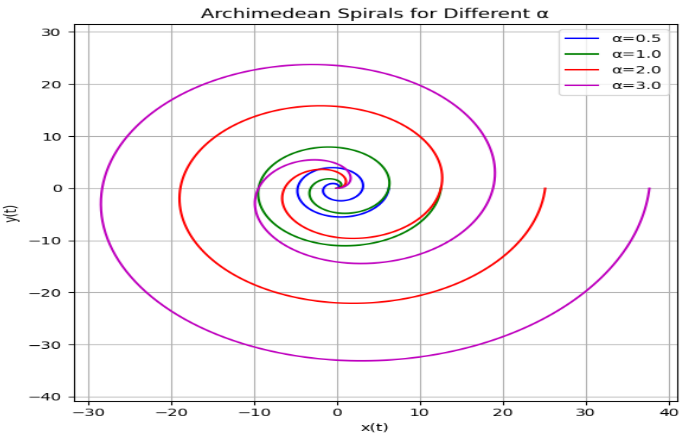

Figure 5.Response function ofequation (3.8) for different value of α

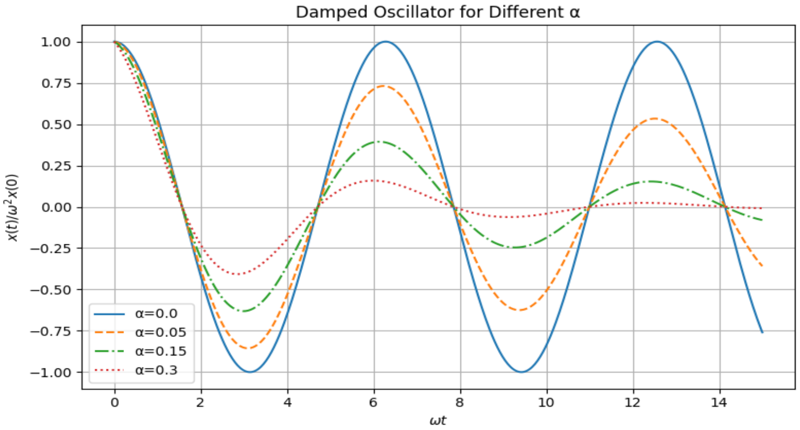

Figure 6.Response function ofequation (3.8) for different value of α

This paper introduces the Sumudu transform method as an effective and efficient technique for solving systems of fractional oscillators subjected to external forces. The method yields solutions expressed as infinite series with easily computable terms. The examples provided demonstrate strong agreement between these results and those obtained via Mittag-Leffler functions. Analytical expressions for the response functions under various forcing types and for values of α within the interval (1 <

$\alpha \leq 2$) have been derived. The findings indicate that the behavior of the driven fractional oscillator closely parallels that of a damped harmonic oscillator. Consequently, the displacement functions effectively characterize dynamics that interpolate between exponential decay when $\alpha = 1$ and pure sinusoidal oscillation when $\alpha = 2$.

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