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Applications Of Second Order Fuzzy Difference Equation In Finance

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ABSTRACT

Fuzzy difference equation growing rapidly developed for the many years. Now the problem is that the solution procedure of difference equation and fuzzy difference are not same. In this paper, we have considered the applications of fuzzy difference equation in finance. Specially use of fuzzy difference equation in predicting future value is calculated and we have verified our results by taking the suitable examples.

Keywords: *Difference equation, fuzzy difference equation, present value, future value*

1. INTRODUCTION

In mathematics, theory of difference equations and its applications play an important role. These equations are used in modeling of many situations such as in population dynamics, probability models, computer science, control engineering, statistical problems, stochastic time series, economics and engineering.

Nonetheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may have a phenomena often called appearance of "ghost" solutions or existence of chaotic orbits that can only happen for higher order differential equations.

A q-th order linear difference equation (synonymously, a linear recurrence relation) is a set of equations of the form

$$x_n - \left(a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + \cdots + a_{n-q}x_{n-q}\right) = r_n \quad (1)$$

For n = q, q + 1, ...

If $r_n = 0$, for all n, the equation is said to be a homogeneous difference equation

otherwise it is non homogeneous difference equation. The term r_n is called the forcing factor. Now if $a_i(i = 1,2,\dots,n)$do not depend on n then the equation said to have constant, coefficients.

2. FUZZY DIFFERENCE EQUATIONS

In 1965, Zadeh initiated the development of the modified set theory known as fuzzy set theory, which is a tool that makes possible description of vague notions and manipulations with them. The basic idea of the fuzzy set theory is simple and natural. The fuzzy set is a function from a set into a lattice or as a special case, into the interval [0,1]. Using it, one can model the meaning of vague notions and also some kinds of human reasoning. The fuzzy set theory and its applications have been extensively developed since the seventies and the industrial interest in fuzzy control has dramatically increased since 1990. There are several books dealing with these aspects [1–8].

A difference fuzzy equation is a fuzzy difference equation when (i) initial condition is fuzzy number, (ii) coefficients is fuzzy number, (iii) initial conditions and coefficients are both fuzzy numbers. Fuzzy difference equation growing rapidly developed for the many years. Now the problem is that the solution procedure of difference equation and fuzzy difference are not same. To study the behavior and solutions of a fuzzy difference equation we need to study the concepts of fuzzy difference, since the fuzzy difference is not same as crisp difference. We can show that every fuzzy difference can converted to system of fuzzy difference equations.

Consider the second order non homogeneous difference equation as

$a_0 u_{n+2} + a_1 u_{n+1} + a_2 u_n = f(n)$ (2)

1.
2. With initial condition $u_{n=0} = u_0$ and $u_{n=1} = u_1$
3. The above second order difference equation is called fuzzy difference equation if any of one case is followed by the above difference equation:

1. The initial condition or conditions are fuzzy number (Type I)
2. The coefficient or coefficients are fuzzy number (Type II)
3. The initial condition or conditions and coefficient or coefficients are fuzzy numbers (Type III)

2.1 Methods for Solving second order fuzzy difference equations

4. Let us consider the second order fuzzy difference equation of Type I:
5. $y(k+2) + ay(k+1) + by(k) = g(k)$ (3)
6. Basically there are three methods to solve second order fuzzy difference equations:

- Classical solution
- Extension Principle solution
- Interval arithmetic solution

3. CLASSICAL SOLUTION

The classical solution of equation(3) is denoted by $\bar{Y}_c(k)$. Let the Ω-cuts of $\bar{Y}_c(k)$ be $[y_1(k,\Omega), y_2(k,\Omega)]$, $k = 0,1,2,3 \ldots$ and $0 \leq \Omega \leq 1$. By substituting these intervals in (3) we get

$y_i(k+2,\Omega) + ay_i(k+1,\Omega) + by_i(k,\Omega) = \mathrm{g(k)}$ (4)

Assuming a and b are positive and $i = 1,2$ subjected to the initial conditions:

$y_1(0,\Omega) = \gamma_{01}(\Omega)$ (5)

$y_1(1,\Omega) = \gamma_{11}(\Omega)$ (6)

$y_2(0,\Omega) = \gamma_{02}(\Omega)$ (7)

$y_2(1,\Omega) = \gamma_{12}(\Omega)$ (8)

where $\bar{\gamma_0} = [\gamma_{01}(\Omega), \gamma_{02}(\Omega)], \bar{\gamma_1} = [\gamma_{11}(\Omega), \gamma_{12}(\Omega)]$. $\bar{Y}_c(k)$ is a solution when the intervals $[y_1(k,\Omega), y_2(k,\Omega)]$, define a fuzzy number for each $k = 0, 1, 2, \cdots$

Similarly, the method for solving second order fuzzy difference equations by extension principal and arithmetic solutions can be found in [5]. Recently in [7] have developed the Lagrange's multiplier method to solve second order linear fuzzy difference equation.

In this section we first consider the elementary concepts, in the mathematics of finance, future value, present value and regular annuities. In all cases the cash amounts, interest rates and number of compounding's may all be fuzzy. Then we look at two methods of comparing fuzzy net cash flows in order to rank fuzzy investment alternatives from

best to worst. For other discussions of the mathematics of finance we refer the reader to ([1], [2], [8], [13], [14] to [27]). This chapter is based on ([3], [5], [6], [7]), and we will be using both triangular and trapezoidal (shaped) fuzzy numbers. Let us explain our procedure for fuzzifying the elementary mathematics of finance. We first write down the

mathematical expression for the problem in finance. Then we substitute fuzzy numbers for some, or all, the parameters in the expression. If in the fuzzy equation we need to solve for the value of some variable, we then solve using a-cuts and interval arithmetic producing the classical solution (if it exists). For example in the fuzzy equation $\bar{A}\bar{X} = \bar{B}$.

we first solve for $\bar{X}$, given $\bar{A}$ and $\bar{B}$, giving $\overline{X_c}$ for the classical solution. If $\overline{X_c}$ fails to exist, we then fuzzify the crisp solution giving solutions $\bar{X}_e$ and $\bar{X}_I$. In, $\bar{A}\bar{X} = \bar{B}$ when $\overline{X_c}$ does not exist, we fuzzify $x = {}^{b}/_{a}$ to get, $\bar{X} = \bar{B} / \bar{A}$. If we evaluate ${}^{B}/_{A}$ using the extension principle we obtain $\bar{X}_e$ and $\bar{X}_I$ which computes ${}^{B}/_{A}$ using Ω-cuts and interval arithmetic.

The other possibility, after fuzzifying the original financial expression, is that all we need to do is evaluate it. Consider $\bar{X} = \bar{A}(1+\bar{B})^n$ for fuzzy numbers $\bar{A}, \bar{B}$, and positive integer n. Given $\bar{A}, \bar{B}$ and n all we need to do is compute X. This can be done in two ways: (1) using the extension principle producing $\bar{X}_e$ or (2) by Ω-cuts and interval arithmetic giving $\bar{X}_I$.

In the first case we can get $\overline{X_c}$, $\bar{X}_e$ and $\bar{X}_I$ and we expect $\overline{X_c} \leq \bar{X}_e \leq \bar{X}_I$. In the second case we have $\bar{X}_e$ and $\bar{X}_I$ with usually

$\bar{X}_e \leq \bar{X}_I$. We prefer $\bar{X}_e$ to $\bar{X}_I$ and only use $\bar{X}_I$ to approximate $\bar{X}_e$ when it is very difficult to obtain $\bar{X}_e$. If we need to compute $\overline{X_c}$, it is the preferred answer, when it exists.

3.1 Future Value

Assume an amount A is invested today at rate r per period for n periods. If S is the amount in the account after n periods, then $S = A(1 + r)^n$. Throughout this chapter we are dealing with compound interest. Interest rates are usually quoted as some percentage per year and then converted to the correct decimal rate per interest period. For example, 9% per year compounded monthly becomes (0.09)/12 = 0.0075 per month. We will assume the r is the interest rate, as a decimal, per interest period. So, A = $1000 at 9% compounded monthly for 4 years produces S = 1000(1 + 0.0075)48 , since n = 48 = 12 · 4 is the number of compounding's in 4 years.

We first fuzzify the compound interest formula by substituting $\bar{A}$ for A and $\bar{r}$ for v. We know $\bar{A} \leq 0$ and the support of $\bar{r}$ will be in [0, 1]. The interest rate v may or may not be known exactly over the n periods so we can model it as a trapezoidal fuzzy number. For example, r = (0.0059/0.0067, 0.0075/0.0083) means the rate is approximately between 8 and 9% compounded monthly. The amount invested is usually known so $\bar{A}$ could be a crisp number. However, we will use a trapezoidal (or triangular) fuzzy number for $\bar{A}$. We wish to compute $\bar{S}$ where,

$\bar{S} = \bar{A}$ (1 + r)n (9)

Let $\bar{A}$ = (a1/a2, a3/a4), r = (r1/r2, r3/r4) with $\bar{A}$ [Ω] = [a1(Ω), a2(Ω)], $\bar{r}$ [Ω] = [r1(Ω)], r2(Ω)].

Before we compute $\bar{S}$ in equation (1) let us justify the fuzzy compound interest expression in (1).

At the end of the first period we have $\bar{A} + \bar{A}\bar{r} = \bar{A}(1 + \bar{r})$ because, for positive fuzzy numbers, multiplication distributes over addition. After two periods

$$\bar{S} = \bar{A}(1+\bar{v}) + \bar{A}(1+\bar{v})\,\bar{v} \qquad (10)$$

or after factoring

$$\bar{S} = \bar{A}(1+\bar{v})^2 \qquad (11)$$

Hence, equation (1) is correct for fuzzy numbers. There are two methods to evaluate equation (1). Using extension principle we obtain $\bar{S}_e$ whose Ω-cuts are

$$S_{e1}(\Omega) = \min a(1+v)^n \mid a \in \bar{A}[\Omega], v \in \bar{v}[\Omega] \qquad (12)$$

$$S_{e2}(\Omega) = \max a(1+v)^n \mid a \in \bar{A}[\Omega], v \in \bar{v}[\Omega] \qquad (13)$$

Clearly, the expression $a(1 + v)^n$ is increasing in both a and v so that

$$S_{e1}(\Omega) = a_1(\Omega)(1 + v_1(\Omega))^n \qquad (14)$$

$$S_{e2}(\Omega) = a_2(\Omega)(1 + v_2(\Omega))^n \qquad (15)$$

$0 \leq \Omega \leq 1$. The second method is to use a-cuts and interval arithmetic defining $\bar{S}_I$. So

$$\bar{S}_I[\Omega] = \bar{A}[\Omega](1+\bar{v}[\Omega])^n \qquad (16)$$

and $\bar{S}_I = \bar{S}_e$ because all intervals are non-negative.

Now we fuzzify by substituting $\bar{A}$ for A, $\bar{r}$ for v and also $\bar{n}$ for n. The fuzzy number of periods n will be a non-negative discrete fuzzy set. That is, there are positive integers ni,

1 ≤ i ≤ K, for some positive integer K, and λi ∈ (0, 1], 1 ≤ i ≤ K, so that

$$\bar{n}(x) = \begin{cases} \lambda_i, & if\ x = n_i \\ 0, & otherwise \end{cases} \qquad (17)$$

The termination of the investment is uncertain and is modeled by $\bar{n}$. An interpretation of λ_i is "the possibility that n = ni is Ai", 1 ≤ i ≤ K. Using this "possibility" interpretation we then need Ai = 1 for some i.

Use the extension principle to compute $\bar{S}_e$ in

$$\bar{S}_e = \bar{A}(1 + \bar{v})^{\bar{n}} \qquad (18)$$

We will not attempt to calculate $\bar{S}_I$ now because Ω-cuts of n will not be intervals but are subsets of $\lambda_1, ..., \lambda_K$. In this case we may not get $\bar{S}_e \leq \bar{S}_I$ and $\bar{S}_I$ is therefore not an approximation to $\bar{S}_e$. Since $\bar{S}_e$ is not difficult to compute (see the next theorem) we will only find S_e in this case.

We only use equations like (4) and (5) when the expression to be evaluated is continuous in all its variables. The expression

$a(1 + r)_n$ is now discrete in n. The membership function for $\bar{S}_e$ is

$$\bar{S}_e(x)=\max\{\pi(a,v,n)|a(1+v)^n=x\} \quad (19)$$

where

$$\pi(a,v,n)=\min\{\bar{A}(a), \bar{v}(v), \bar{n}(n)\} \quad (20)$$

There is an easier way to get $\bar{S}_e$. Let $\bar{S}_{ni} = \bar{A}(1 + r)^{ni}$, $1 \le i \le K$.

Theorem 1. $\bar{S}_e(x) = \max_{1\le i\le K}\{\min\{\bar{S}_{ni}(x), \lambda_i\}\}$

Proof. Let the right hand side of the above equation be Γ(x).

1. We first show $\bar{S}_e(x) \le \Gamma(x)$. Assume $\bar{S}_e(x) = \gamma$. There exists a, r, n = n_i, so that $\pi(a, r, n_i) = \gamma$, $a(1+r)^{ni} = x$ and $\bar{A}(a) \ge \gamma$, $\bar{r}(r) \ge \gamma$ and $\lambda_i \ge \gamma$. Using n = n_i, let us call $\bar{S}_e$ by the name $\bar{S}_{ni}$. From equations (4) and (4·5) we get $x \in \bar{S}_{ni}[\gamma]$. So the minimum of $\bar{S}_{ni}(x)$ and λ_i is greater than, or equal to γ. Hence, $\Gamma(x) \ge \gamma = \bar{S}_e(x)$.

2. Now show $\Gamma(x) \le \bar{S}_e(x)$. Let $\Gamma(x) = \gamma$. There is an i between 1 and K so that, $\min\{\bar{S}_{ni}(x), \lambda_i\} = \gamma$. Hence $\bar{S}_{ni}(x) \ge \gamma$ and $\lambda_i \ge \gamma$. So $x \in \bar{S}_{ni}[\gamma]$ and there is an $a \in \bar{A}[\gamma]$, $r \in \bar{r}[\gamma]$ so that $x = a(1 + r)^{ni}$. This means $\pi(a, r, n_i) \ge \gamma$ and $a(1 + v)^{ni} = x$. Hence $\bar{S}_e(x) \ge \gamma$.

Theorem 4.1 allows us to easily compute $\bar{A}(1 + \bar{r})^{\bar{n}}$. We first find the $\bar{S}_{ni}$ as in equations (6), (7) or equation (8), $1 \le i \le K$. Cut each $\bar{S}_{ni}$ off at height λ_i, and then take the max of the resulting fuzzy sets.

Example 1.1

Let $\bar{A}$ = (90/100, 100/110) and $\bar{r}$ = (0.07/0.08, 0.09/0.10) with n given as follows: (1) $_{n1} = 4$, $\lambda_1 = 0.6$; (2) $n_2 = 5$, $\lambda_2 = 1.0$; and (3) n = 6, λ = 0.5. $\bar{S}_e$ is shown in Figure 4.1. The individual trapezoidal shaped fuzzy numbers $\bar{S}_{ni}$, i = 1, 2, 3, disappear when aggregated into $\bar{S}_e$. $\bar{S}_e$ may not be a fuzzy number.

3.2 Present Value:

We wish to find the present value of a future amount *A*, *n* periods in the future, if *v* is the interest rate per period. Let *S* be the present value of *A*. There are two ways to compute *S*. The first method is that the present value of *A* equals S_1, if you can invest S_I today, at rate *v* per period, so that in *n* periods it accumulates to *A*. That is, S_1 solves

$$S_1(1 + y)^n = A \quad (21)$$

The second method is to solve equation (13) for S_1, giving

$$S_2 = A(1 + y)^{-n} \quad (22)$$

In non-fuzzy mathematics $S_1 = S_2$, however, the two methods may produce different results for fuzzy mathematics.

We now substitute $\bar{A}$ for A and $\bar{r}$ for v and solve $\bar{S}_1$ and $\bar{S}_2$. Quite often in finance future cash amounts *A* are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain a value for *A*. Fuzzy mathematics allows an alternative to having to use an exact value for A. $\bar{A}$ will be a trapezoidal fuzzy number $(a_1/a_2, a_3/a_4)$ meaning that the future amount is approximately between a_2 and a_3.

We first solve for $\bar{S}_1$ and we may obtain three solutions S_{1c}, S_{1e} and S_{1I}.

The classical solution solves

$$\bar{S}_{1c}(1 + \bar{v})^n = \bar{A} \quad (23)$$

Taking Ω-cuts we obtain

$$[s_{1c1}(\Omega), s_{1c2}(\Omega)][w_1(\Omega), w_2(\Omega)] = [a_1(\Omega), a_2(\Omega)] \quad (24)$$

Where

$$w_i(\Omega) = (1 + v_i(\Omega))^n \quad (25)$$

$i = 1, 2$. Now $\bar{A} \ge 0$, so $\bar{S}_{1c} \ge 0$. We get

$$S_{1c1}(\Omega) = \frac{a_1(\Omega)}{(1 + v(\Omega))^n} \quad (26)$$

$$S_{1c2}(\Omega) = \frac{a_2(\Omega)}{(1 + v(\Omega))^n} \quad (27)$$

The "interval" $[s_{1c1}(\Omega), s_{1c2}(\Omega)]$ may, or may not, define a fuzzy number $\bar{S}_{1c}$

Continuing with $\bar{S}_1$ the solution $\bar{S}_{1e}$ fuzzifies $a/(1 + v)^n$. Alfa-cuts of $\bar{S}_{1e}$ are

$S_{1e1}(\Omega) = \min\{\frac{a}{(1+v)^n} \,|a \in \bar{A}[\Omega], r \in \bar{v}[\Omega]$ (28)

$S_{1e2}(\Omega) = \max\{\frac{a}{(1+v)^n} \,|a \in \bar{A}[\Omega], r \in \bar{v}[\Omega]$ (29)

$S_{1e2}(\Omega) = \left[\frac{a_1(\Omega)}{(1+v_2(\Omega))^n}, \frac{a_2(\Omega)}{(1+v_3(\Omega))^n}\right]$ (30)

Lastly, $\bar{S}_{1I}[\Omega]$ comes from $\frac{\bar{A}(\Omega)}{(1+v[\Omega])^n}$ (31)

after substituting the intervals for $\bar{A}(\Omega)$ and $\bar{v}[\Omega]$ and simplifying using interval arithmetic. We see that $\bar{S}_{1e} = \bar{S}_{1I}$

Turning to $\bar{S}_1$, we have only $\bar{S}_{2e}$ and $\bar{S}_{2I}$. $\bar{S}_{2e} = \bar{A}(1+\bar{v})^n$ using the extension principle. Hence, $\bar{S}_{2e} = \bar{S}_{1e}$. Also , $\bar{S}_{1e} = \bar{S}_{1I}$is just equation (23). Therefore, $\bar{S}_{1e} = \bar{S}_{1I} = \bar{S}_{2e} = \bar{S}_{2I}$. $\bar{S}_{1c}$, if it exists, is the only solution that always satisfies equation (13), for $\bar{A}$ and $\bar{v}$ submitted for A and v, evaluated by Ω-cuts and interval arithmetic.

Since $\bar{S}_{1c}$ satisfies the present value equation our solution strategy is: (1) choose $\bar{S}_{1e}$ when it exists; and (2) are $\bar{S}_{1e} = \bar{S}_{2e}$ when $\bar{S}_{1c}$ does not exist.

Example 1.2

Let $\bar{A} = (8000/10000, 12000/14000)$ and $\bar{v} = (0.05/0.06, 0.06/0.07)$. If n=10, then $\bar{S}_{1c}$ exists since

$S_{1c1}(\Omega) = \frac{(8000+2000\Omega)}{(1.05+0.01\Omega)^{10}}$ (32)

$S_{1c2}(\Omega) = \frac{(14000-2000\Omega)}{(1.07-0.01\Omega)^{10}}$ (33)

$0 \leq \Omega \leq 1$, defines the Ω-cuts of a trapezoidal shaped fuzzy number.

Now we allow n to be a positive discrete fuzzy set n, since the number of interest periods into the future is uncertain. We have two equations to consider for present value of $\bar{A}$.

$\bar{S}_1(1+\bar{v})^{\bar{n}} = \bar{A}$ (34) And

$\bar{S}_2 = \frac{\bar{A}}{(1+\bar{v})^{\bar{n}}}$ (35)

From equations (13) and (14). We will not use equation(26) for two reasons: (1) $\bar{S}_{1c}$ may not exist; and (2) more importantly $(1+\bar{v})^{\bar{n}}$ may not be a fuzzy number (Example 1.1) so that Ω- cuts can be a union of intervals and then we cannot use standard interval arithmetic. When Ω- cuts of $(1+\bar{v})^{\bar{n}}$ are not a single interval, we cannot solve equation (26) for the a-cuts of SIc. So, we only have S2e to consider. We know Sle = S2e and we do not calculate $\bar{S}_{1e}$ and $\bar{S}_{2e}$ (no intervals for Ω-cuts of $\bar{n}$). From the extension principle

$\bar{S}_{2e}(x) = \max\{\pi(a,r,n)\} \,|a(1+r)^{-n} = x\}$, (36)

For π given by equation (12). If we write $\bar{S}_n = \bar{A}(1+\bar{v})^{i\bar{n}}$, then

Theorem 2. $\bar{S}_{2e}(x) = max_{1\leq i\leq K}\{\min\{\bar{S}_{ni}(x), \lambda_i\}\}$

Proof: Same as Theorem 1.1.

$\bar{S}_{2e}$ need not be a fuzzy number.

3.3 Annuities

7. We consider only ordinary (regular) annuities where: (1) the payment period equals the interest period; and (2) the equal periodic payments *A* are at the end of each period for *n* periods. We first look at the future value of an annuity and then the present value.
8.
9. **3.3.1 Future Value**
10. The future value S of the regular annuity of n equal payments A at rate *v* per period is

$S = A(1+v)^{\bar{n}1} + (1+v)^{\bar{n}2} + \cdots \dots \dots \dots \dots + A(1+r) + A$ (37)

Or

$$S = Aq(n, r) \quad (38)$$

Where, $q(n,r) = \frac{(1+v)^n - 1}{r}$ (39)

Fuzzifying equation (29) we wish to find S where

$\bar{S} = \bar{A}(1+\bar{v})^{\bar{n}1} + \bar{A}(1+\bar{v})^{\bar{n}2} + \cdots \ldots \ldots + \bar{A}(1+\bar{v}) + \bar{A}$ (40)

There are two solutions $\bar{S}_e$ and $\bar{S}_I$. $\bar{S}_e$ is based on the extension principle and its Ω-cuts are

$S_{e1}(\Omega) = \min\{aq(n,r))|a \in \bar{A}[\Omega], \mathrm{r} \in \bar{v}[\Omega]\}$ (41)

$S_{e1}(\Omega) = \max\{aq(n,r))|a \in \bar{A}[\Omega], \mathrm{r} \in \bar{v}[\Omega]\}$ (42)

Since S is an increasing function of a and v we obtain

$\bar{S}_e[\Omega] = [a_1(\Omega)\mathrm{q}(\mathrm{n}, r_1(\Omega)), a_1(\Omega)q(n, r_2(\Omega))]$ (43)

$0 \le \Omega \le 1$. $\bar{A} \ge 0$ and $\bar{v}$ is in [0,1].

Now $\bar{S}_I$ will not equal $\bar{S}_e$. $\bar{S}_I[\Omega]$ is $A[\Omega]\mathrm{q}(\mathrm{n}, \bar{v}[\Omega])$ and we obtain Ω-cut of $\bar{S}_I$ as

$a_1(\Omega)\frac{(1+v_1(\Omega)^n - 1}{r_2(\Omega)}, a_2(\Omega)\frac{(1+v_2(\Omega)^n - 1}{r_1(\Omega)}$ (44)

Clearly, $\bar{S}_e \le \bar{S}_I$. We will use $\bar{S}_e$ as the solution.

Let n = $\bar{n}$, the termination date is fuzzy. Let $\bar{S}_e$ be $\bar{S}_{ni}$; when we use n = n_i in equation (32), $0 \le i \le K$. Then we obtain the result, as in Theorem 1.1 and 1.2

$\bar{S}_e = max_{1 \le i \le K}\{\min\{\bar{S}_{ni}(x), \lambda_i\}\}$ (45)

Example: 1.

Let $\bar{A} = (150/190, 200/220)$, $\bar{v} = (0.06/0.09, 0.10/0.11)$ and n = n_1 =10 for $\lambda_1 = 0.8$, n = n_1 =12 for $\lambda_1 = 1.0$, n = n_3 =14 for $\lambda_1 = 0.6$. Thus future value of this fuzzy annuity $\bar{S}_e$ is shown in Figure 4.2. It is not a fuzzy number.

3.3.2 Present Value

The present value S of a regular annuity is

$S = A(1+v)^{-1} + A(1+v)^{-2} + \cdots + A(1+v)^{-n}$ (46)

or

$S = A\beta(n,r)$ (47)

For $\beta(n,r) = \frac{1-(1+v)^{-1}}{r}$ (48)

The present value of a fuzzy annuity is

$\bar{S} = \bar{A}(1+\bar{v})^{-1} + \bar{A}(1+\bar{v})^{-2} + \cdots + \bar{A}(1+\bar{v})^{-n}$ (49)

We simplify fuzzify equation (39), Alpha-cuts of $\bar{S}_e$ are

$S_{e1} = \min\{a\beta(n,r)|a \in)|a \in \bar{A}[\Omega], \mathrm{r} \in \bar{v}[\Omega]\}$ (50)

$S_{e2} = \max\{a\beta(n,r)|a \in)|a \in \bar{A}[\Omega], \mathrm{r} \in \bar{v}[\Omega]\}$ (51)

Now S is an increasing function of A and a decreasing function of v so

$\bar{S}_e[\Omega] = [a_1(\Omega)\beta(\mathrm{n}, r_2(\Omega)), a_2(\Omega)\beta(n, r_1(\Omega))]$ (52)

$0 \le \Omega \le 1$. We again $\bar{S}_e \subset \bar{S}_I$ and $\bar{S}_e \ne \bar{S}_I$. It is a good, and short, exercise to evaluate

$\bar{A}[\Omega]\beta(\mathrm{n}, \bar{V}(\Omega))$ (53)

Which is $\bar{S}_I[\Omega]$ and see $\bar{S}_e \subsetneq \bar{S}_I$. We suggest $\bar{S}_e$ as the present value.

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