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Solution Of Three-Dimensional Helmholtz Equation By Using Triple Laplace Transform

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ABSTRACT

The Helmholtz partial differential equation has various applications in the different fields such as electromagnetics, quantum mechanics, engineering, physics and its mathematical models power the technologies that are crucial in neonatal diagnostics, imaging, simulation, and intervention planning making it an essential tool in the computational background of advanced neonatal care. We present a novel approach to solving the Helmholtz equation using Triple Laplace Transform. An inversion of triple Laplace transforms has been achieved numerically by employing the Brancik technique. Numerical results are represented by graphically.

Keywords: *Helmholtz equation; Triple Laplace Transform; Inverse Triple Laplace Transform; Partial differential equations.*

1. INTRODUCTION

The Helmholtz equation, a fundamental partial differential equation in mathematical physics, plays a pivotal role in modeling wave propagation and diffusion processes. In the context of neonatal surgery, where precision and minimal invasiveness are critical, the Helmholtz equation underpins several advanced medical technologies. Its applications are particularly significant in enhancing imaging modalities such as ultrasound and diffuse optical tomography, which are essential for diagnosing and monitoring conditions in neonates. Furthermore, it contributes to computational modeling for brain activity monitoring, acoustic simulations, and surgical planning. By enabling accurate simulation of tissue interactions and wave behaviors, the Helmholtz framework supports improved diagnostic accuracy and safer, more effective neonatal surgical interventions. This paper explores these interdisciplinary applications, highlighting the equation's role in bridging mathematical theory and clinical practice in neonatal care. Atangana [1] obtained solution of Mboctara equation by using Triple Laplace Transform (TLT). The paper presents various properties and theorems related to this new operator and demonstrates its utility by solving a specific class of Mboctara differential equations. Khan et al. [2] solved the fractional order two-dimensional heat problem using the TLT. They introduced the method of the TLT to solve a class of fractional partial differential equations. They specifically apply this method to the two-dimensional fractional-order homogeneous heat equation, utilizing the Caputo fractional derivative. The paper demonstrates the application of this transform to obtain analytical solutions for the heat equation under certain initial and boundary conditions. Numerical plots are provided to illustrate the behavior of the solutions.

Juraev et al. [3] solved the Helmholtz equation for the Cauchy problem and matrix factorizations in different spaces. The authors discuss its significance in various physical contexts, including seismology, electromagnetic radiation, and acoustics. In the study of water waves, the equation showed the transition to more general scattering problems. Alkhalifah et al. [4] derived the wavefield solutions using machine-learning functions that were subject to Helmholtz equation constraints. This research introduces a novel approach that leverages neural networks to solve the Helmholtz equation in wave propagation studies. By incorporating the equation into the loss function, the neural network is trained to predict wavefield values at specific spatial locations. The network utilizes automatic differentiation to compute partial derivatives, ensuring adherence to the Helmholtz equation. The authors demonstrate the effectiveness of this method through experiments on models such as a two-box-shaped scatterer and highlighting the potential of neural networks in geophysical wavefield modeling.

H. Cheng and M. Peng [5] established the solution of Helmholtz equation in three dimensions by employing the Improved Element Free Galerkin (IEFG) Method. The authors employed the Moving Least Squares approximation to construct trial functions, which enhances computational efficiency and accuracy. The study further explores the impact of weight functions, influence domain scale parameters, node distribution, and penalty factors on solution accuracy. Numerical results demonstrated that the proposed IEFG method not only accelerates computation compared to the traditional Element-Free Galerkin (EFG) method. Chai et al. [6] obtained the solution of Helmholtz equation by using the Extrinsic Enriched Finite Element Method (EFEM). To enhance numerical performance, the authors propose an extrinsic EFEM that enriches the standard linear approximation by trigonometric functions. This enrichment is realized using the partition of unity, effectively capturing the oscillatory nature of solutions associated with higher wave numbers.

G. Bao et al. [7] examined the pollution effect and the viability of using the discrete singular convolution (DSC) technique and a local spectral method to obtain the solution of Helmholtz equation. S. Gong et al. [8] explored decomposition techniques for the Helmholtz problem using parallel overlapping Schwarz domains. P. Roland and S. Olivier [9] discretize the Helmholtz equation using finite differences in space and solved by using Stochastic Galerkin method. Brancik [6] developed the method of numerically inverting 3D NILT using a complicated Fourier series approximation. This research is crucial for solving complex problems in engineering and physics. The proposed technique utilizes complex Fourier series combined with the quotient-difference algorithm to achieve accurate and efficient inversion of 3D Laplace transforms. This approach enhances the computational efficiency and accuracy of solving multidimensional problems involving Laplace transforms.

2. BASIC EQUATIONS

Definition 1: The TLT of f (x, y, z) is defined as [1]

$$L_3[f(x,y,z)] = F(q,s,a) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(qx+sy+az)} f(x,y,z)dx\,dy\,dz, \tag{1}$$

provided that the integral exists and q, s, a are complex numbers.

Definition 2: The Inverse TLT is defined as [1]

$$L_3^{-1}\{f(q,s,a)\} = f(x,y,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{qx}dq \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{sy}ds \frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty} e^{az}F(q,s,a)\,da, \tag{2}$$

where $f(q,s,a)$ must be analytic for all q, s, a in the region defined by $\text{Re}\,(q) \geq a, \text{Re}\,(s) \geq b, \text{Re}\,(a) \geq c$ for some a, b, c are constants to be chosen suitably.

Theorem: If $u(x,y,z)$ is any continuous function for $x, y, z \geq 0$ & $L_3[u(x,y,z)] = U(q,s,a)$. Then by [2]

$$L_xL_yL_z\left[\frac{\partial^n}{\partial z^n}u(x,y,z)\right] = a^nU(q,s,a) - \sum_{i=0}^{n-1} a^{n-i-1}\frac{\partial^i}{\partial z^i}U(q,s,0), \tag{3}$$

$$L_xL_zL_y\left[\frac{\partial^n}{\partial y^n}u(x,y,z)\right] = s^nU(q,s,a) - \sum_{i=0}^{n-1} s^{n-i-1}\frac{\partial^i}{\partial z^i}U(q,0,a), \tag{4}$$

$$L_zL_yL_x\left[\frac{\partial^n}{\partial x^n}u(x,y,z)\right] = q^nU(q,s,a) - \sum_{i=0}^{n-1} q^{n-i-1}\frac{\partial^i}{\partial z^i}U(0,s,a). \tag{5}$$

The Helmholtz equation [3] is given by

$$(\nabla^2 + k^2)u = 0, \tag{6}$$

where ∇^2 is the Laplacian operator and k is the wave number.

Equation (6) can be extended into three-dimensional Helmholtz equation as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2u(x,y,z) = 0, \tag{7}$$

where, $k = \dfrac{2\pi}{\lambda}, \lambda$ is wavelength.

3. GENERAL PROCEDURE

The general method for applying the TLT to solve the three-dimensional Helmholtz Equation is presented in this section.

Consider equation (7) as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2u(x,y,z) = 0, \tag{8}$$

Applying TLT on equation (8), we obtain

$$q^2U(q,s,a) - qU(0,s,a) - \frac{\partial}{\partial x}U(0,s,a) + s^2U(q,s,a) - sU(q,0,a) - \frac{\partial}{\partial y}U(q,0,a) + a^2U(q,s,a)$$

$-aU(q,s,0) - \frac{\partial}{\partial z}U(q,s,0) + k^2U(q,s,a) = 0,$

$[q^2 + s^2 + a^2 + k^2]\,U(q,s,a) = qU(0,s,a) + \frac{\partial}{\partial x}U(0,s,a) + sU(q,0,a) + \frac{\partial}{\partial y}U(q,0,a) + aU(q,s,0) + \frac{\partial}{\partial z}U(q,s,0),$

$$U(q,s,a) = \frac{qU(0,s,a)+\frac{\partial}{\partial x}U(0,s,a)+sU(q,0,a)+\frac{\partial}{\partial y}U(q,0,a)+aU(q,s,0)+\frac{\partial}{\partial z}U(q,s,0)}{q^2+s^2+a^2+k^2}, \quad (9)$$

Utilizing Inverse TLT on equation (9), we obtain

$$u(x,y,z) = L_3^{-1}\left[\frac{qU(0,s,a)+\frac{\partial}{\partial x}U(0,s,a)+sU(q,0,a)+\frac{\partial}{\partial y}U(q,0,a)+aU(q,s,0)+\frac{\partial}{\partial z}U(q,s,0)}{q^2+s^2+a^2+k^2}\right], \quad (10)$$

Equation (9) represents the general solution in Laplace domain. To find an original function, an inversion of the TLT of the solution obtained in equation (10) has been performed numerically by employing Brancik (2010) technique.

4. ILLUSTRATIVE EXAMPLES

The section focuses on providing basic examples to demonstrate the general procedure for the problems. We present graphs to demonstrate the computational analysis of the relevant examples.

EXAMPLE 1:

Consider $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2u(x,y,z) = 0, \quad (11)$$

subject to following initial conditions

$$U(x,y,0) = xy,\ U(x,0,z) = xz,\ U(0,y,z) = yz,$$

$$\frac{\partial}{\partial x}U(0,s,a) = \frac{\partial}{\partial y}U(q,0,a) = \frac{\partial}{\partial z}U(q,s,0) = 0. \quad (12)$$

Applying TLT on equation (11) and using equations (3) – (5), we obtain

$$[q^2 + s^2 + a^2 + k^2]\,U(q,s,a) = qU(0,s,a) + \frac{\partial}{\partial x}U(0,s,a) + sU(q,0,a) + \frac{\partial}{\partial y}U(q,0,a) + aU(q,s,0)$$

$+\frac{\partial}{\partial z}U(q,s,0),$

$[q^2 + s^2 + a^2 + k^2]\,U(q,s,a) = qU(0,s,a) + sU(q,0,a) + aU(q,s,0),$

$[q^2 + s^2 + a^2 + k^2]\,U(q,s,a) = \frac{1}{s^2a^2} + \frac{1}{q^2a^2} + \frac{1}{q^2s^2},$

$$U(q,s,a) = \frac{q^2+s^2+a^2}{q^2s^2a^2[q^2+s^2+a^2+k^2]}. \quad (13)$$

Utilizing inverse TLT on equation (13), we obtain

$$u(x,y,z) = L_3^{-1}\left[\frac{q^2+s^2+a^2}{q^2s^2a^2[q^2+s^2+a^2+k^2]}\right]. \quad (14)$$

Equation (13) represents the general solution in the Laplace domain. An inversion of the TLT of the solution found in equation (14) has been carried out numerically using the Brancik [10] technique in order to identify the solution in the original domain.

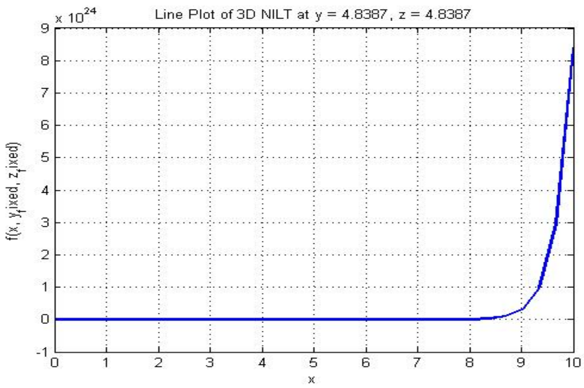

Figure 1: Solution of equation (14) by taking y = z = 4.8387

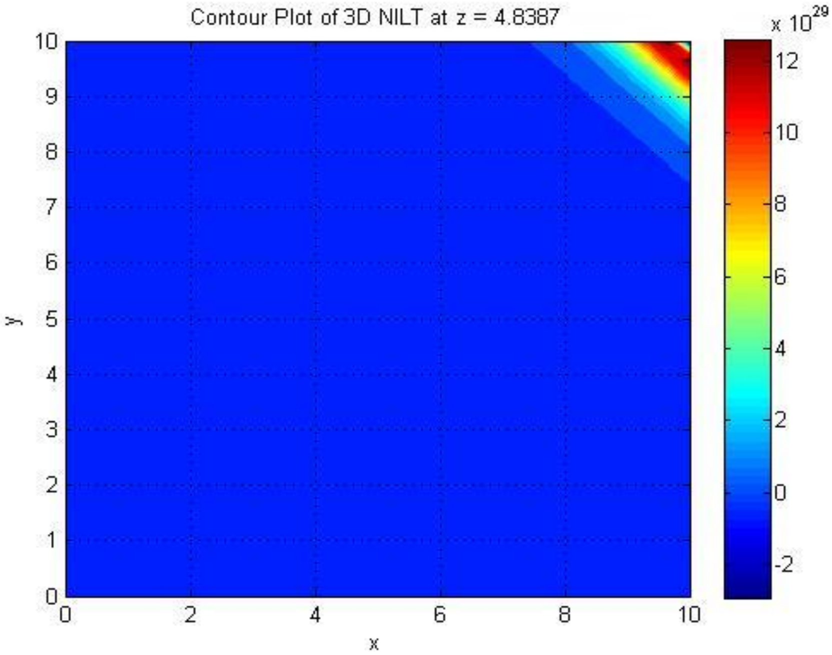

Figure 2: Contour graph of equation (14) by taking z = 4.8387

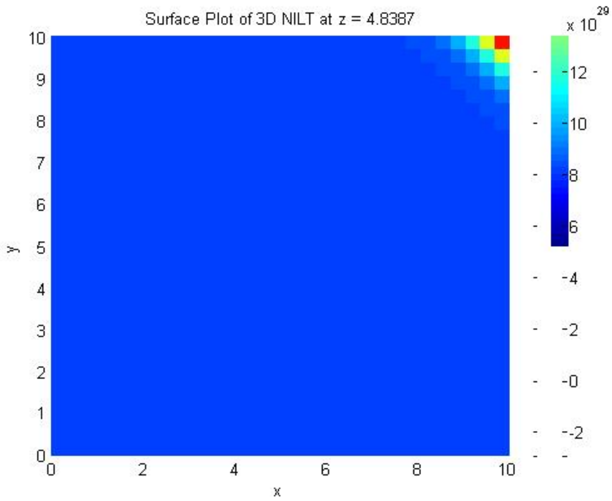

Figure 3: Surface plot of equation (14) by taking z = 4.8387

EXAMPLE 2:

Consider

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+k^2 u(x,y,z)=0, \tag{15}$$

subject to following initial conditions

$$U\,(x,y,0)=\sin(x)\sin(y)\,, U(x,0,z)=\sin(x)\sin\,(z), U(0,y,z)=\sin(y)\sin\,(z),$$
$$\frac{\partial}{\partial x}U(0,s,a)=\frac{\partial}{\partial y}U(q,0,a)=\frac{\partial}{\partial z}U(q,s,0)=0. \tag{16}$$

Applying TLT on equation (15) and using equations (3) – (5), we obtain

$$[q^2+s^2+a^2+k^2]\,U(q,s,a)=qU(0,s,a)+\frac{\partial}{\partial x}U(0,s,a)+sU(q,0,a)+\frac{\partial}{\partial y}U(q,0,a)+aU(q,s,0)+\frac{\partial}{\partial z}U(q,s,0),$$

$[q^2 + s^2 + a^2 + k^2]\, U(q, s, a) = qU(0, s, a) + sU(q, 0, a) + aU(q, s, 0),$

$[q^2 + s^2 + a^2 + k^2]\, U(q, s, a) = \frac{1}{(s^2+1)(a^2+1)} + \frac{1}{(q^2+1)(a^2+1)} + \frac{1}{(q^2+1)(s^2+1)},$

$$U(q, s, a) = \frac{q^2+s^2+a^2+3}{(q^2+1)(s^2+1)(a^2+1)[q^2+s^2+a^2+k^2]}. \quad (17)$$

If we take wave number $\lambda = 3.6276$ then $k^2 = 3$.

Hence equation (17) becomes,

$U(q, s, a) = \frac{q^2+s^2+a^2+3}{(q^2+1)(s^2+1)(a^2+1)[q^2+s^2+a^2+3]},$

$$U(q, s, a) = \frac{1}{(q^2+1)(s^2+1)(a^2+1)}. \quad (18)$$

Equation (18) represents the general solution in Laplace domain. By applying inverse TLT on equation (18), we obtain

$$u(x, y, z) = L_3^{-1}\left[\frac{1}{(q^2+1)(s^2+1)(a^2+1)}\right], \quad (19)$$

$$u(x, y, z) = \sin(x)\sin(y)\sin(z). \quad (20)$$

The nature of obtained solution i.e., equation (20) is presented graphically in figures 3 to 6.

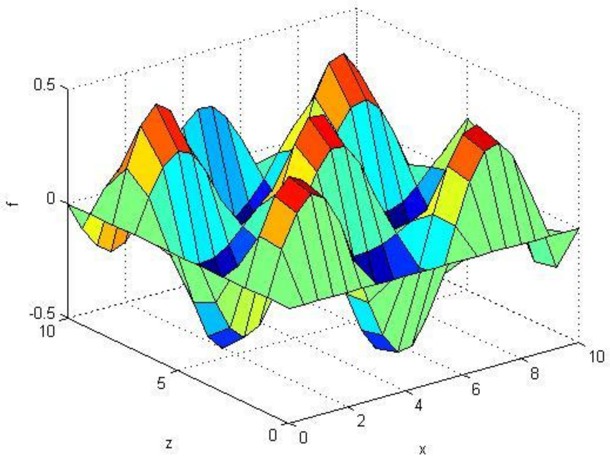

Figure 4: 3D plot of solution of equation (20)

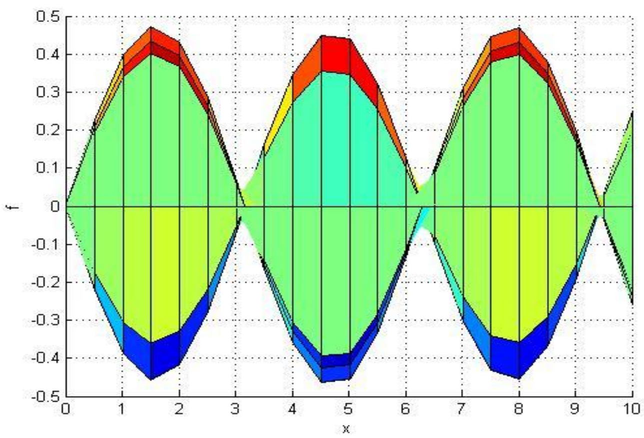

Figure 5: 2D plot of solution of equation (20) by taking y = z = 0.5

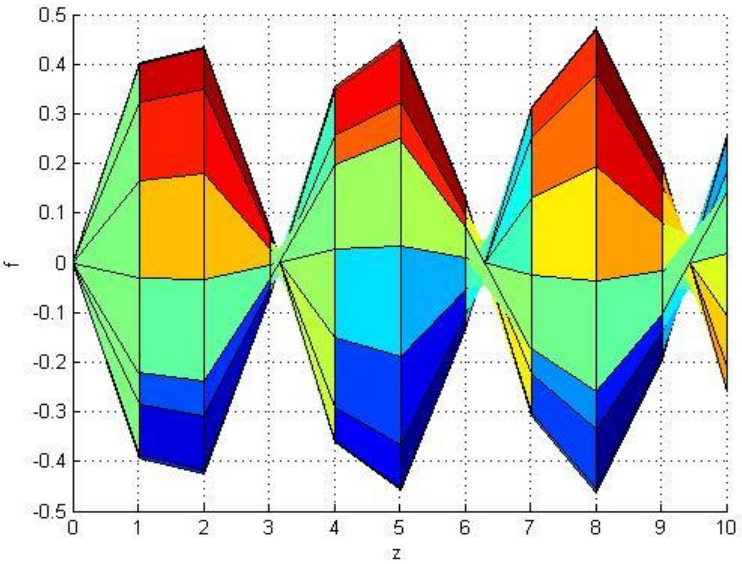

Figure 6: 2D surface plot of solution of equation (20) by taking y = z = 0.5

5. CONCLUSIONS

We have solved three-dimensional Helmholtz equation that arises in various fields such as electromagnetics, quantum mechanics, engineering and physics in modeling wave propagation and diffusion processes. First, the Triple Laplace and Inverse Triple Laplace transforms were used to determine the solution in Laplace domain and original domain. Additionally, the suggested method's approximation strategy has been applied to get the numerical solutions. Numerical and graphical results validate the efficiency of the proposed approach.

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