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On the Lattice of Subnormal Subgroups of Matrix Groups

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ABSTRACT

In this paper we give the lattice structure of the lattice of subnormal subgroups of the group of2 x 2 matrices over Z_P, having determinant value 1, under matrix multiplication modulo p, where is a prime number and p=2,3. The properties satisfied are modular, dually semi modular and consistent

WealsointroducetheconceptofalmostsubnormalsubgroupsofagroupGandwegivethelatticestructureofthelatticeof almost subnormal subgroups of the above-mentioned group when p= 2, 3, 5 and 7.

The motivation for this study is the paper ‘Consistent Dually Semi modular lattices’ by Karen M. Gragg and Joseph P.S Kung,Journal of Combinatorial Theory,SeriesA60,246-263(1992)

Keywords: *Subnormal subgroups, Almost subnormal sub groups.*

1. INTRODUCTION

The study on the lattices of sub groups of a given group was started with the work of Richard Dedekind’s [9]in1877. After thatithaswitnessedsomanydevelopmentsbythecontributionofmanyauthors.

In 1992 Karen M. Gragg and P.S Kung [6] have attempted to characterize the lattice groups with a consistent lattice of subgroups.Inthatendeavortheydiscoveredthatthelatticeofsubnormalsubgroupsofafinitegroupisconsistentanddually (lower) semi modular.A.Vethamanickam has cited from their theorem and has given a counter example in his thesis [8].

In 2015 D.Jebaraj Thiraviam [7] has worked on the lattice of subgroups of a group G of 2 x 2 matrices over Zp having determinant value 1 under matrix multiplication modulo p where p is prime. Then he investigated some weaker properties like consistency , super solvable , semi modular, 0- semi modular , 0 – modular , 0 – distributive and 0- super modularity when p = 2, 3, 5 and 7.

In this paper in section 2 we give the lattice structure of the lattice W (G) of subnormal subgroups of G when p= 2 and 3. Also we verify the properties consistency, dually (lower) semi modular and modular in W (G).

In section 3 we introduce the concept almost subnormal subgroups of G and we give the lattice structure, properties in the lattice AW (G) of the subnormal subgroups of G.

Definition 1.1[1]-A partial order on a non-empty set P is a binary relation ≤ on P that is reflexive, anti symmetric and transitive. The pair (P, ≤) is called *a* ***partially ordered set or poset***. Poset (P, ≤) is ***totally ordered*** if every x, y ∈ P are comparable, that is x ≤ y or y ≤ x.

Anon –emptysubsetSofPisa***chain***inPifSistotallyorderedby≤.

Definition1.2[1]-Let(P,≤)beaposetandletS ⊆ P.An***upperbound*** forSisanelementx∈ Pforwhichs≤x∀ s∈ S.The leastupperboundofSiscalledthe ***supremum***or***join***ofS.A***lowerbound***ofSisanelementx∈Pforwhichx≤s∀s∈S.

The greatest lower bound of Siscalled the ***infimum*** or ***meet***of S. Poset (P, ≤) iscalled a ***lattice*** if everypair x,y ∈ P hasa supremum and an infimum.

Definition1.3[2]-AlatticeLiscalled ***semimodular***ifwheneveracoversa∧b,thena∨bcoversb,foralla,b∈L.

Definition1.6[10]-Intheposet (P,≤),acoversborbiscoveredbya(innotation,a ≻borb ≺ a)ifandonlyifb<aandfor no x∈ p, b < x <a. An element ‘a’ is an ***atom***, if a≻0 and a ***dual atom***, if a≺1.

Definition1.7[3]-AlatticeLissaidtobe***modularlattice***,ifa∨(b∧c)=(a∨b)∧cforeverya,b,cinLanda≤c.

Definition1.8[6]-A lattice is ***dually semi modular*** for lower semi modular if for all x ∨ y covers y implies y coversx∧ y. An element j in L is a **join *irreducible*** if j=a ∨ b=a ⇒j =a or j= b or equivalently j covers at most one element.

Definition1.9[6]-ALatticeLissaidtobe***consistent***ifforalljisajoin-irreduciblejandallelementsxinL,j∨xisjoin irreducibleintheupperinterval[x,1].

Definition 1.10[3]-A lattice L issaid to be **super *solvable***, if it containsa maximal chaincalled an M-chain in which every element is modular. By a modular element m in a lattice L, we mean x ∨ (m ∧ y) = (x ∨ m) ∧y whenever x ≤ y in L.

Definition1.11[4]-AlatticeLissaidtobe***0-distributive*** ifforallx,y,z∈L,wheneverx∧y=0andx∧z=0thenx∧(y ∨z) =0.

Definition1.12[4]-A lattice Lissaid tobe ***0-modular*** ifwheneverx≤ yandy∧z =0,then x=(x∨z)∧y, for allx,y,z∈ L.

Definition1.13[5]-AsubgroupNofGissaidtobea***normalsubgroup***ofGifforeveryg∈Gandn ∈N,gng^{-1}∈N.

Definition1.14[11]-ThelatticeLwith0satisfiesthe***generaldisjointnessproperty***ifx∧y=0and(x∨y)∧z=0implyx ∧(y ∨z)=0.

2. SUBNORMALSUBGROUPS[6]

A subgroup H of G is said to be ***subnormal*** if there exists a finite chain of subgroups H = $G_0 \subseteq G_1 \subseteq \ldots\ldots \subseteq G_n$= G such that $G_i$is normal in G_{i+1}. The lattice of subnormal subgroups of G is denoted by W (G)

Whenp=2

{e}isanormalsubgroupofL_1,H_1,H_2,H_3.L_1isanormalsubgroupofG.

LatticestructureofW(G)whenp=2

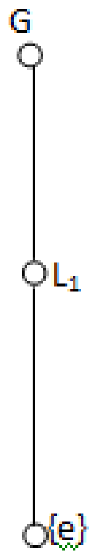

fig1

Whenp=3-{e}isnormalin H_1, K_1, K_2, K_3, K_4.H_1isnormalin M_1, M_2, M_3, M_4.K_1isnormalin M_1.K_2 isnormalin M_2.$K_3$is normal in M_3.$K_4$is normal in M_4.L_1, L_2, $L_3$are normal in N_1. $N_1$is normal in G, M_1, M_2, M_3, M_4 are not normal in G.

LatticestructureofW(G)whenp=3

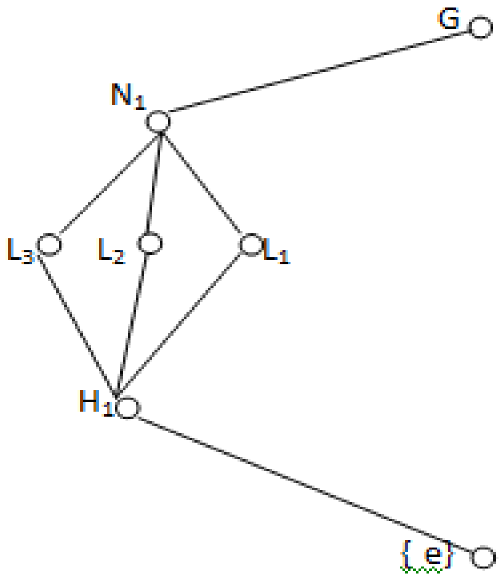

fig2

Lemma2.1-Whenp≤3,wherepisprime,W(G)isconsistent,dually(lower)semimodular,modular.

Proof:Fromfig1andfig2wecansaythatwheneverjisajoin –irreducibleinW(G),thenx ∨ jisjoin-irreducibleinthe upper interval [x , 1] , for every x ϵ W(G) when p ≤ 3.Therefore, W (G) is consistent when p ≤ 3.

Fromfig1and2wecansaythat,forallx,yϵW(G),x∨ycoversyimpliesycoversx∧y.Therefore,W(G)isdually (lower)semimodularwhenp≤3.

Fromfig1and2,wecansaythatforalltheelementsinW(G)i.e.,a,b,c ϵW(G).a∨(b∧c)=(a∨b)∧c.Therefore,W (G) ismodularwhenp≤3.

3. ALMOSTSUBNORMALSUBGROUPS

Let H⊆G. H is called ***almost subnormal subgroup*** of G if there exist a finite chain of subgroups

$H\subseteq G_1\subseteq G_2\subseteq G_3\subseteq G_k$where$G_k$isamaximalsubgroupofGsuchthat$G_i$isnormalinG_{i+1}foralli=1,2,k-1.AW(G)denotesthelatticeofalmost

LatticestructureofAW(G)whenp=2

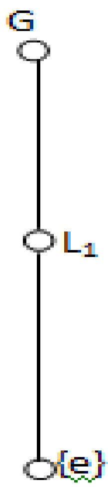

fig3

LatticestructureofAW(G)whenp=3

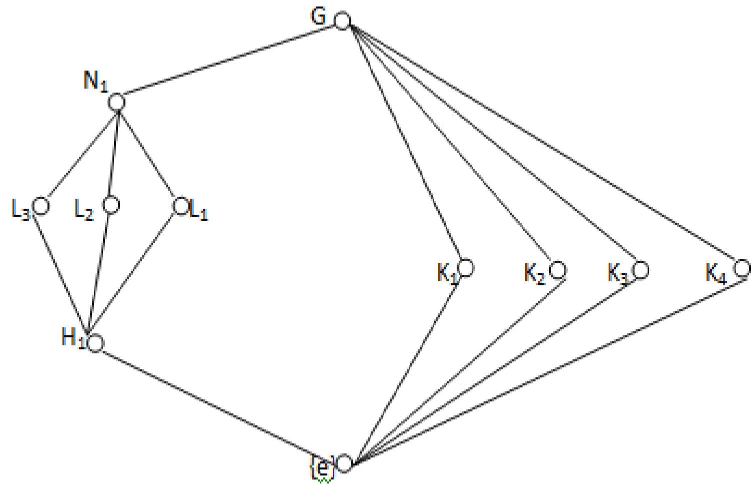

Fig4

When p=5{e} is normal in H_1, K_1toK_{10}, $M_1$to M_6**,** $H_1$is normal in $L_1$to L_{15}, $N_1$to N_{10}, $Q_1$to Q_6, $K_1$to $K_{10}$is normalin $N_1$to N_{10}. M_1 to $M_6$is normalin $Q_1$to Q_6. L_1, L_2,$L_3$is normal in P_1,L_4, L_5,$L_6$is normal in P_2, L_7, L_8,$L_9$is normal in P_3, L_{10}, L_{11},$L_{12}$is normal in P_4,L_{13}, L_{14},$L_{15}$is normal in P_5.N_1 to $N_{10}$is normal in $R_1$to R_{10}. Q_1 to $Q_6$is normal in $T_1$to T_6. . P_1 to $P_5$is normal in $S_1$to S_5.S_1 to $S_5$are not normal in G. T_1 to $T_6$are not normal in G.

LatticestructureofAW(G)whenp=5

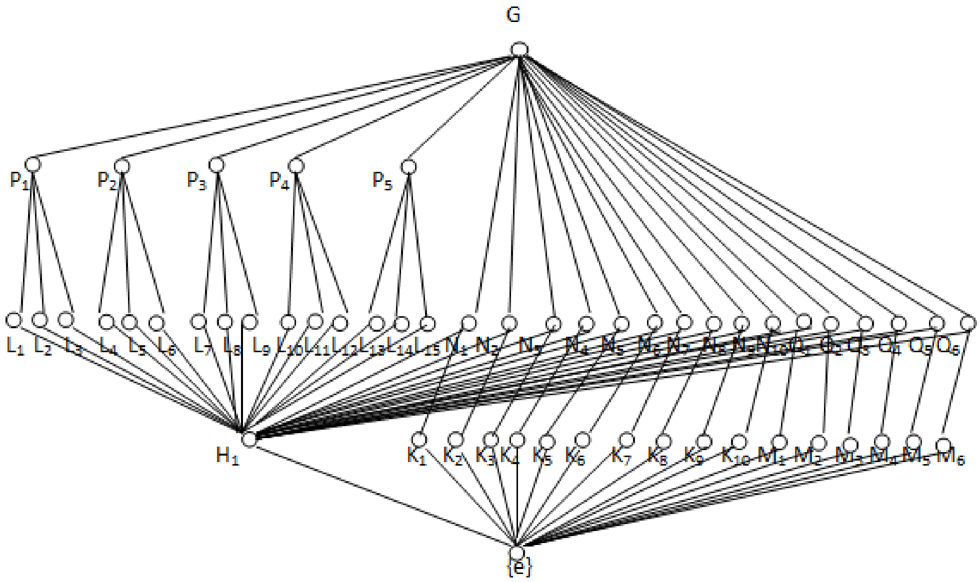

Fig5

Whenp=7{e} is normal in H_1, $K_1$to K_{28}, $N_1$to N_8. H_1 is normal in$L_1$to L_{21},$M_1$to M_{28},$R_1$to R_8.$K_1$to $K_{28}$is normal in$M_1$to M_{28}. $L_1$to $L_{21}$is normal in $P_1$to P_{21}. $M_1$to $M_{28}$is normalin $Q_1$to Q_{28}. $P_1$to $P_{21}$is normal in $S_1$to S_{21}. $N_1$to $N_8$is normal in $T_1$to T_{28}, $R_1$to R_8. $T_1$to T_8isnormalin $U_1$to U_8. $N_1$to N_8isnormalin $U_1$to U_8. $R_1$to R_8isnormalin $U_1$to U_8. $S_1$to $S_{21}$are not normal in G. $U_1$to $U_8$are not normal in G

LatticestructureofAW(G)whenp=7

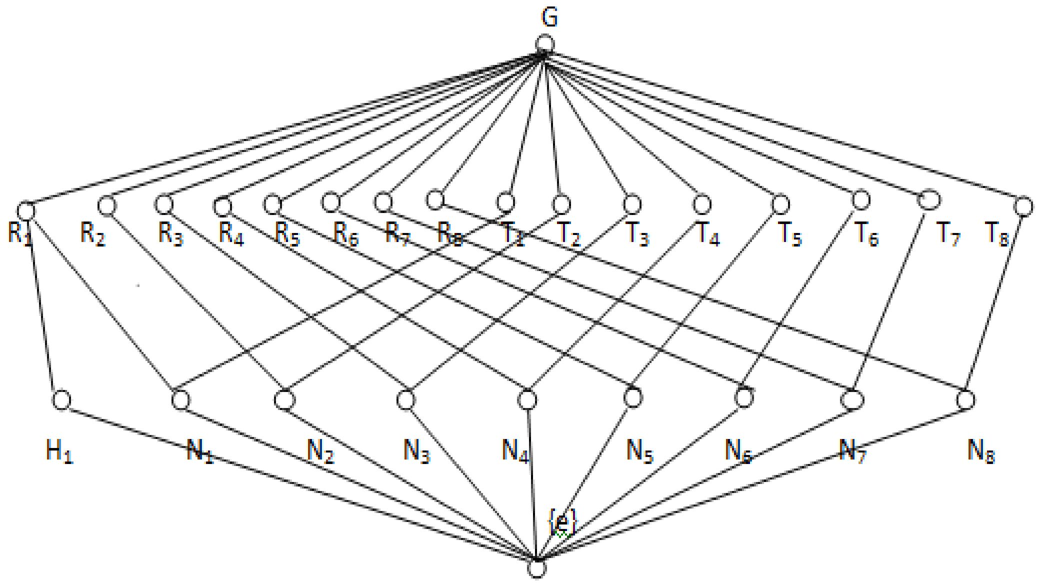

Fig6

Lemma3.1-Whenp=3,5,7AW(G)isnotduallysemimodular,notsemimodular,not0–distributiveandnot0–modular.

Proof:

Whenp=3Fromfig4,$L_1 \vee K_1=G$□$L_1$and$L_1 \wedge K_1=\{e\}$□$K_{1.}$ThereforeAW(G)isnotduallysemimodular.$L_1 \wedge K_1$ $=\{e\}$□L_1 but$L_1 \vee K_1=G$□K_1ThereforeAW(G)isnotsemimodular. For $H_1 \leq K_1$,$H_1 \vee (K_1 \wedge K_2)=K_1and(H_1 \vee K_1) \wedge K_2=K_2$ThereforeAW(G) isnotmodular.$H_1 \wedge K_1=\{e\}$,$H_1 \wedge K_2=\{e\}$and$H_1 \wedge (K_1 \vee K_2)=H_1 \neq \{e\}$Therefore AW(G) is not 0-distributive .For $H_1 \leq L_1$, $L_1 \wedge K_1= \{e\}$ and $(H_1 \vee K_1) \wedge L_1 \neq H_{1.}$Therefore AW(G) is not 0- modular. ***Whenp=5***Fromfig5,$P_1 \vee P_2=G$□P_1but$P_1 \wedge P_2=H_1$□P_2ThereforeAW(G)isnotduallysemimodular.$K_1 \wedge K_2= \{e\}$,K_1□$\{e\}$but$K_1 \vee K_2=G$□K_2.ThereforeAW(G)isnotsemimodular.For$N_1 \leq P_2$,$N_1 \vee (N_2 \wedge P_2)=N_1but(N_1 \vee N_2) \wedge P_2= P_2$Therefore AW (G) is not modular.$H_1 \wedge K_1 = \{e\}$, $H_1 \wedge K_2= \{e\}$ and $H_1 \wedge (K_1 \vee K_2) = H_1 \wedge G = H_1 \neq \{e\}$. Therefore AW (G) isnot0- distributive. For $H_1 \leq N_1$, $N_1 \wedge K_1= \{e\}$ but $(H_1 \vee K_1) \wedge N_1 =N_1 \wedge N_1 \neq H_1$ThereforeAW (G) is not 0- modular.

Whenp=7Fromfig6,$T_3 \vee T_4=G$□T_3but$T_3 \wedge T_4=\{e\}$□T_4.ThereforeAW(G)isnotduallysemimodular.

$N_1 \wedge N_2=\{e\}$□N_1but $N_1 \vee N_2=G$□N_2.ThereforeAW(G)isnot semimodular.For$N_1 \leq T_1$, $N_1 \vee (N_2 \wedge T_1)= N_1 \vee \{e\}=N_1$ and$(N_1 \vee N_2) \wedge T_1=T_1$ThereforeAW(G)isnotmodular.

$H_1 \wedge N_1=\{e\}$,$H_1 \wedge N_2=\{e\}$but$H_1 \wedge (N_1 \vee N_2)=H_1 \wedge G=H_1 \neq \{e\}$ThereforeAW(G)isnot0-distributive.For$N_1 \leq T_1$,$T_1 \wedge N_2=\{e\}$but$(N_1 \vee N_2) \wedge T_1 =G \wedge T_1 \neq N_{1.}$ThereforeAW(G)isnot0-modular.

Lemma3.2-Whenp=2and3thegeneraldisjointnesspropertyissatisfiedinAW(G).

Proof: When p= 2 and 3 AW (G) satisfies the general disjointness property since there is no choice of three elements x,y and z.

Lemma3.3-Whenp=5and7thegeneraldisjointnessconditionfailsinAW(G).

Proof: Fromfig$H_1 \wedge K_1=0$,$(H_1 \vee K_1) \wedge K_2=0$.but$H_1 \wedge (K_1 \vee K_2) \neq$Fromfig6,$H_1 \wedge N_1=0$,$(H_1 \vee N_1) \wedge N_2=0$ but$H_1 \wedge (N1 \vee N_2) \neq 0$ Therefore the general disjointness condition fails in AW (G) when p =5 and 7.

Lemma 3.4 **-**When $p \leq 3$ W (G) and AW (G) are super solvable.

Proof: Fromfig3,$\{e\} \subseteq L_1 \subseteq G$istheM-chaininwhicheveryelementismodularFromfig4andfig5,$\{e\} \subseteq H_1 \subseteq L_1 \subseteq N_1 \subseteq G$istheM-chaininwhicheveryelementismodular.ThereforeW(G)andAW(G)aresupersolvableforp$\leq$ 3.

3.CONCLUSION

In this paper we have given the lattice structure of the lattice of subnormal subgroups of G when p= 2 and 3 .We also verified that W (G) is consistent, dually semi modular and modular. We have also introduced almost subnormal subgroups and have given the lattice structure of AW(G) and verified the properties like not dually semi modular, not semi modular , not modular, not 0- distributive, not 0- modular, general disjointness property and super solvable.

CONFLICTSOFINTEREST

"Therearenoconflictstodeclare."

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