**Journal of Neonatal Surgery**
*ISSN(Online): 2226-0439*
*Vol. 13, (2024)*
https://www.jneonatalsurg.com
OPEN ACCESS

# A Note on an Abstract Nonlinear delay differential Equation with nonlocal condition ia A new three steps iteration

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Cite this paper as: Rupesh T. More ,Pankaj M. Sonawane, (2024) A Note on an Abstract Nonlinear delay differential Equation with nonlocal condition ia A new three steps iteration. *Journal of Neonatal Surgery*, 13, 1103-1120.

**ABSTRACT**
In this paper, we study the existence, uniqueness and other properties of solutions of an abstract nonlinear delay differential equation with nonlocal condition. The tool employed in the analysis is based on application of new three steps iteration. New three steps iteration method has equally important contribution to study various properties such as dependence on initial data, closeness of solutions and dependence on parameters and functions involved therein.A new three steps iteration process introduced by V. Karakaya,Y. Atalan, K. Dogan, and NH. Bouzara [8]..

**Keywords:** Existence ,uniqueness and continuous dependence; abstract differential equation with finite delay ; new three steps iteration method

## 1. INTRODUCTION

The aim of the present paper is to study existence, Osgood type uniqueness and qualitative properties of mild solutions of nonlinear delay differential equation with nonlocal condition. The main tools employed in the analysis are based on the applications new three steps equation Method. Using Tychonov's fixed point theorem, the method of successive approximations, and the comparison method, S. Sugiyama [27] studied the existence and uniqueness of solutions of the following problem:

$$\frac{d\omega(t)}{dt} = f(t,\omega(t),\omega(t-1)), \qquad (1.1)$$

for $0 \le t \le t_1$, with the conditions

$$\omega(t-1) = \phi(t) \quad (0 \le t < 1), \qquad (1.2)$$

$$\omega(0) = \omega_0, \qquad (1.3)$$

where $\omega$ and $f$ represent $n$-dimensional vectors (see [27] for details) and Stokes [28] has discussed the same problems as above for nonlinear differential equations.

From the above works, we can see a fact, although the delay differential equation have been investigated by some authors. However, to our knowledge, the an abstract nonlinear delay differential equation with nonlocal conditions and an infinitesimal generator of operators has not been discussed extensively. So motivated by all the works above, the aim of this paper is to prove the existence and uniqueness of solutions of the delay differential equation of the form

$$\omega'(t) + A\omega(t) = f(t,\omega(t),\omega(t-1)), \qquad (1.4)$$

for $t \in J = [0,b], (b > 0)$ under the conditions

$$\omega(t-1) = \phi(t) \quad (0 \le t < 1) \qquad (1.5)$$

$$\omega(0) + g(\omega) = \omega_0, \qquad (1.6)$$

where $-A$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in $X$, $f \in C(J \times X \times X, X)$, $g \in C(C(J,X),X)$ and $\phi(t)$ is a continuous function for $0 \le t < 1$, $\lim_{t\to 1-0}\phi(t)$ exists, for which we denote by $\phi(1-0) = c_0$. If we consider the solutions of (1.4) for $t \in J$, we obtain a function $\omega(t-1)$ which is unable to define as solution for $0 \le t < 1$. Hence, we have to impose some condition, for example the condition (1.5). We note that, if $0 \le t < 1$, the problem is reduced to delay differential equation

$$\omega'(t) + A\omega(t) = f(t,\omega(t),\phi(t)),$$

with initial condition $\omega(0) + g(\omega) = \omega_0$. Here, it is essential to obtain the solutions of (1.4)–(1.6) for $0 \leq t \leq b$, so that, we suppose in the sequel $b$ is not less than 1.

The development of integral and differential equations and their advantages in science and applied mathematics are largely attributed to the notion of fixed points theory of iterative approximation(TIA) (see [22, 23, 24,26,29]). . In this regard,for specific classes of operators, a lot of researchers has found several iteration techniques in terms of their convergence, rate of convergence, equivalence of convergence etc.see ( [12, 25]). A number of mathematicians have studied the challenges of existence, uniqueness, and other properties of solutions of particular forms of IVP (1.4)-(1.6) and its variants under a set of hypotheses via various tactics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and some of the references cited therein [14, 15, 16, 17, 18, 19, 20, 21, 26].

Our main objective here is to investigate the global existence of solution to (1.4)-(1.6) The chapter is organized as follows, we present the preliminaries and hypotheses. Section 2.3 deals with existence and uniqueness of the solutions., we discuss result on continuous dependency on initial data on function ,on parameter.

**Three Steps Iteration Process:**

In 2017, V. Karakaya and Y. Atalan, and NH. Bouzara [8] introduced the following three steps iteration process:

$$\begin{cases} \omega_{k+1} &= Ty_k, \\ y_k &= (1-\xi_k)z_k + \xi_k T z_k, \\ z_k &= T\omega_k, \ k \in \mathbb{N} \cup \{0\}. \end{cases} \qquad (1.7)$$

with the real control sequence $\{\xi_k\}_0^\infty \in [0, 1]$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$.

**Definition 1.1** *([8], p.627)* Let $(X, d)$ be a complete metric space and $T: X \to X$ be a weak-contraction for which there exist $\delta \in (0,1)$ and $L \geq 0$ such that

$$\| T\omega - Ty \| \leq \delta \| \omega - y \| + L \| y - Ty \|. \quad (1.8)$$

Then,$T$ has a unique fixed point.

**Theorem 1.2** *(([8], p.627)) Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T: C \to C$ be a weak-contraction map satisfying condition (1.8). Let $\{\omega_k\}_{k=0}^\infty$ be an iterative sequence generated by the scheme (1.7) with a real control sequence $\{\xi_k\}_{k=0}^\infty$ in $[0,1]$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$. Then $\{\omega_k\}_{k=0}^\infty$ converges to a unique point $\omega^*$ of $T$.*

**Lemma 1** *([12], p.4) Let $\{\beta_k\}_{k=0}^\infty$ be a nonnegative sequence for which one assumes there exists $k_0 \in \mathbb{N} \cup \{0\}$, such that for all $k \geq k_0$ one has satisfied the inequality*

$$\beta_{k+1} \leq (1-\mu_k)\beta_k + \mu_k \gamma_k, \qquad (1.9)$$

*where $\mu_k \in (0,1)$, for all $k \in \mathbb{N} \cup \{0\}$, $\sum_{k=0}^{\infty} \mu_k = \infty$ and $\gamma_k \geq 0, \forall k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds*

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k. \qquad (1.10)$$

Before proceeding to the statement of our main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let $X$ be the Banach space with norm $\|\cdot\|$. Let $B = C(J, X)$ be the space of all continuous functions from $J$ into $X$ endowed with the supremum norm

$$\| \omega \|_B = \sup\{\| \omega(t) \| : t \in J\}.$$

We list the following hypotheses:

**H1)** $-A$ is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in $X$ such that

$\| T(t) \| \leq \mathcal{M},\ t \geq 0$ , for all $t \in J$.

**H2)** There exist $\mathcal{G} > 0$ such that

$\| g(\omega) - g(\overline{\omega}) \| \leq \mathcal{G} \| \omega - \overline{\omega} \|$ for $\omega, \overline{\omega} \in B$

**H3).** The function $f: J \times X \times X \to X$ is continuous and there exist constant $\mathcal{L} > 0$ in (1.4) satisfies the condition

$\| f(t,\omega,y) - f(t,\overline{\omega},\overline{y}) \| \leq \mathcal{L}(\| \omega - \overline{\omega} \| + \| y - \overline{y} \|)$ for $\omega, \overline{\omega}, y, \overline{y} \in X$

**H4)** $g$: B → X and there exist $\mathcal{G} > 0$ such that

$\| g(\omega) - g(\overline{\omega}) \| \leq \mathcal{G} \| \omega - \overline{\omega} \|$ , for $\omega, \overline{\omega} \in B$

**H5)**There exists constant $\alpha > 0$ such that

$\| \omega(s) - \overline{\omega}(s) \| + \| \omega(s-1) - \overline{\omega}(s-1) \| \leq 2\alpha \| \omega - \overline{\omega} \|, \omega, \overline{\omega} \in B, s \in J$

**Definition 1.3** Let $-A$ is the infinitesimal generator of a $C_0$ −semigroup $T(t), t \geq 0$, on a Banach space $X$. The function $\omega \in B$ given by

$$\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^t T(t-s) f(s,\omega(s),\phi(s))ds, \quad (1.11)$$

for $0 \leq t < 1$, and

$$\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^1 T(t-s) f(s,\omega(s),\phi(s))ds$$
$$+ \int_1^t T(t-s) f(s,\omega(s),\omega(s-1))ds, \quad (1.12)$$

for $1 \leq t \leq b$, is called the mild solution of the problem (1.4)-(1.6).

## 2 Existence And Uniqueness of Solutions Via New Three Steps Iterative Method

Now, we are able to state and prove the following main theorem which deals with the existence and uniqueness of solutions of the problem (1.4)-(1.6) We first prove the main conclusion..

**Theorem 2.1** *Assume that hypotheses* $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ *hold. Let* $\{\xi_k\}_{k=0}^{\infty}$ *be real sequence in [0,1] satisfying* $\sum_{k=0}^{\infty} \xi_k = \infty$ *Then the equation* (1.1)-(1.2) *has a unique solution* $\omega \in C[t_0, b]$ *which is the required solution and is obtained by the three steps iterative method starting with any element* $\omega_0 \in X$ *Moreover, if* $\omega_k$ *is the k th successive approximation, then one has*

$$\| \omega_{k+1} - \omega \| \leq \frac{\Omega^{2k+2}}{e^{(1-\Omega)\sum_{i=0}^{k} \xi_i}} \| \omega_0 - \omega \|, \quad (2.1)$$

*where* $\Omega = max\{\Omega_1, \Omega_2\} = max\{(\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}b), (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}b + \mathcal{M}\mathcal{L} + 2\alpha\mathcal{M}\mathcal{L}b)\}$ *and* $\Omega < 1$.

,

**Proof 2.2** Let $\omega(t) \in B$ and define the operator We define the operator

**Case 1** for $0 \leq t < 1$

$$F\omega(t) = T(t)[\omega_0 - g(x)]$$
$$+ \int_0^t T(t-s) f(s,\omega(s),\phi(s))ds, \quad (2.2)$$

and **Case 2** for $1 \leq t \leq b$

$$F\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^1 T(t-s) f(s,\omega(s),\phi(s))ds$$
$$+ \int_1^t T(t-s) f(s,\omega(s),\omega(s-1))ds, \quad (2.3)$$

Let $\{\xi_k\}_{k=0}^{\infty}$ be iterative sequence generated by new three steps iteration method (1.7) for the operator given in (2.2-2.3) with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in [0, 1]. We will show that $\omega_k \to \omega$ as $k \to \infty$. From (1.7),(2.2-2.3) and assumptions, we obtain

**Case 1** For $0 \le t < 1$

$$\begin{aligned}
&\| z_k(t) - \omega(t) \| \\
&= \| (F\omega_k)(t) - (F\omega)(t) \| \\
&\le \| T(t)[g(\omega_k) - g(\omega)] + \int_0^t T(t-s) \| f(s, \omega_k(s), \phi(s)) \\
&\quad - f(s, \omega(s), \phi(s)) \| ds \\
&\le \mathcal{M}\mathcal{G} \| \omega_k - \omega \| + \mathcal{M} \int_0^t \mathcal{L}(\| \omega_k - \omega \| + \| \phi(s) - \phi(s) \|) ds \\
&\le \mathcal{M}\mathcal{G} \| \omega_k - \omega \| + \mathcal{M}\mathcal{L}b \| \omega_k - \omega \| \\
&\le (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}b) \| \omega_k - \omega \| \\
&\le \Omega_1 \| \omega_k - \omega \| \\
&\le \Omega \| \omega_k - \omega \| \quad (2.4)
\end{aligned}$$

**Case 2** For $1 \le t \le b$

$$\begin{aligned}
&\| z_k(t) - \omega(t) \| \\
&= \| (F\omega_k)(t) - (F\omega)(t) \| \\
&= \mathcal{M}\mathcal{G} \| \omega_k - \omega \| + \mathcal{M} \int_0^1 \| f(s, \omega_k(s), \phi(s)) - f(s, \omega(s), \phi(s)) \| ds \\
&\quad + \mathcal{M} \int_1^t \| f(s, \omega_k(s), \omega_k(s-1)) - f(s, \omega(s), \omega(s-1)) \| ds \\
&\le \mathcal{M}\mathcal{G} \| \omega_k - \omega \| + \mathcal{M}\mathcal{L} \| \omega_k - \omega \| \\
&\qquad + \mathcal{M}\mathcal{L}b(\| \omega_k(s) - \omega(s) \| + \| \omega_k(s-1) - \omega(s-1) \| \\
&\le (\mathcal{M}\mathcal{G} \| \omega_k - \omega \| + (\mathcal{M}\mathcal{L} + \mathcal{M}\mathcal{L}b) \| \omega_k - \omega \| + (2\mathcal{M}\mathcal{L}b\alpha) \| \omega_k - \omega \| \\
&\le (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + \mathcal{M}\mathcal{L}b + 2\alpha\mathcal{M}\mathcal{L}\mathrm{b}) \| \omega_k - \omega \| \\
&\le \Omega_2 \| \omega_k - \omega \| \\
&\le \Omega \| \omega_k - \omega \| \quad (2.5)
\end{aligned}$$

Now by taking supremum in the inequality

$$\| z_k - \omega \|_B \le \Omega \| \omega_k - \omega \|_B \quad (2.6)$$

and

$$\begin{aligned}
&\| y_k(t) - \omega(t) \| = \| (1-\xi_k) z_k(t) + \xi_k (F z_k)(t) - \omega(t) \| \\
&= \| (1-\xi_k) z_k(t) + \xi_k (F z_k)(t) - (1-\xi_k)\omega(t) - \xi_k \omega(t) \| \\
&= \| (1-\xi_k)(z_k(t) - \omega(t)) + \xi_k (F z_k)(t) - (F\omega)(t) \| \\
&\le [(1-\xi_k) \| z_k(t) - \omega(t) \| + \xi_k \| (F z_k)(t) - (F\omega)(t) \|] \quad (2.7)
\end{aligned}$$

Hence, by taking supremum in the inequality (2.7) and then use (2.6) to get

$$\begin{aligned}
&\| y_k - \omega \|_B \le [(1-\xi_k) \| z_k - \omega \|_B + \xi_k \| F z_k - F\omega \|_B] \\
&\quad \le (1-\xi_k) \| z_k - \omega \|_B + \xi_k \| F z_k - F\omega \|_B \\
&\quad \le [1 - \xi_k(1-\Omega)] \| z_k - \omega \|_B \\
&\quad \le \Omega[1 - \xi_k(1-\Omega)] \| \omega_k - \omega \|_B \quad (2.8)
\end{aligned}$$

Therefore, using (2.6) and (2.8), we obtain

$$\begin{aligned}
&\| \omega_{k+1} - \omega \|_B = \| F y_n - \omega \|_B \\
&\le \Omega \| y_k - \omega \|_B
\end{aligned}$$

$\leq \Omega^2[1-\xi_k(1-\Omega)] \parallel \omega_k-\omega \parallel_B$ (2.9)

Thus by induction,we get

$$\parallel \omega_{k+1}-\omega \parallel_B \leq \Omega^{2k+2} \prod_{j=0}^{j=k} [1-\xi_k(1-\Omega)] \parallel \omega_0-\omega \parallel_B \quad (2.10)$$

Since $\xi_k \in [0,1]\ \forall k \in \mathbb{N} \cup \{0\}$ the definition $\Delta$ yields,

$$\Rightarrow \Omega\xi_k < \xi_k \leq 1 \Rightarrow \xi_k(1-\Omega) < 1, \forall k \in \mathbb{N} \cup \{0\} \qquad (2.11)$$

From the classical analysis, we know that $1-x \leq e^{-x}, \forall x \in [0,1]$ Hence by utilizing this fact with (2.11) in (2.10), we obtain

$$\parallel \omega_{k+1}-\omega \parallel_B \leq \Omega^{2k+2} \prod_{k=0}^{k=n} \exp^{-(1-\Omega)} \sum_{j=0}^{k} \xi_j \parallel \omega_0-\omega \parallel_B$$

$$\parallel \omega_{k+1}-\omega \parallel_B \leq \frac{\Omega^{2k+2}}{e^{(1-\Omega)\sum_{i=0}^{k}\xi_i}} \parallel \omega_0-\omega \parallel_B \quad (2.10)$$

above inequality denoted by

Thus, we have proved (2.1). Since $\sum_{k=0}^{\infty} \xi_k = \infty$ , then we have

$$e^{-(1-\Omega)\sum_{i=0}^{k}\xi_i} \to 0 \quad as \quad k \to \infty \quad (2.11)$$

Hence using this, the inequality (2.1) implies $\lim_{k\to\infty} \parallel \omega_{k+1}-\omega \parallel_B = 0$ and therefore, we get $\omega_k \to \omega$ as $k \to \infty$.

□

**Remark** It is an interesting to note that the above inequality (2.1) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the problem (1.4)-(1.6) for $t \in I$

## 3 **Continuous dependency of initial data via New three steps iteration**

Suppose $\omega(t)$ and $\overline{\omega}(t)$ are solutions of (1.4) with initial data

$$\omega(0)+g(\omega)=\omega_0 \qquad (3.1)$$

and

$$\overline{\omega}(0)+g(\overline{\omega})=\overline{\omega}_0 \qquad (3.2)$$

respectively, where $\omega_0, \overline{\omega}_0$ are elements of the space $X$. Then looking at the steps as in the proof of Theorem 2.1 , we define the operator for the equation (1.4) with initial conditions (3.2)

**Case 1** for $0 \leq t < 1$

$$F\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^t T(t-s) f(s,\omega(s),\phi(s))ds, \quad (3.3)$$

Now for **Case 2** for $1 \leq t < b$,

$$F\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^1 T(t-s) f(s,\omega(s),\phi(s))ds + \int_1^t T(t-s) f(s,\omega(s),\omega(s-1))ds, \qquad (3.4)$$

and also define

**Case 1** for $0 \leq t < 1$

$$\overline{F}\overline{\omega}(t) = T(t)[\overline{\omega}_0 - g(\overline{\omega})] + \int_0^t T(t-s) f(s, \overline{\omega}(s), \phi(s)) ds, \quad (3.5)$$

and **Case 2** for $1 \leq t < b$,

$$\overline{F}\overline{\omega}(t) = T(t)[\overline{\omega}_0 - g(\overline{\omega})] + \int_0^1 T(t-s) f(s, \overline{\omega}(s), \phi(s)) ds + \int_1^t T(t-s) f(s, \overline{\omega}(s), \overline{\omega}(s-1)) ds, \quad (3.6)$$

We shall deal with the continuous dependence of solutions of equation (1.1) on initial data.

**Theorem 3.1** *Suppose the function F in equation (1.4) satisfies the hypothesis* $(\mathcal{H}_1)$-$(\mathcal{H}_5)$. *Consider the sequences* $\{\omega_k\}_{k=0}^{\infty}$ *and* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *generated by new three steps iteration method associated with operators F in* (2.2-2.3) *and* $\overline{F}$ *in* (3.5-3.6) *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in [0, 1] satisfying* $\frac{1}{2} \leq \xi_k \; \forall k \in \mathbb{N} \cup \{0\}$. *If the sequence* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *converges to* $\overline{\omega}$, *then we have*

$$\| \omega - \overline{\omega} \|_B \leq \frac{5\mathcal{M}^*}{1-\Delta} \quad (3.7)$$

where

$$\mathcal{M}^* = \mathcal{M} \| \omega_0 - \overline{\omega}_0 \|$$

where $\Delta = \max\{\Delta_1, \Delta_2\}$

$(\mathcal{MG} + \mathcal{ML}b) = \Delta_1$ and $\mathcal{MG} + \mathcal{ML} + 2\mathcal{ML}b\alpha = \Delta_2$ and $\Delta < 1$.

**Proof 3.2** Suppose the sequences $\{\omega_k\}_{k=0}^{\infty}$ and $\{\overline{\omega}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators F in (2.2-2.3) and $\overline{F}$ in (3.5-3.6), respectively with the real control sequence $\{\omega_k\}_{k=0}^{\infty}$ in [0, 1] satisfying $\frac{1}{2} \leq \xi_k \; \forall k \in \mathbb{N} \cup \{0\}$. From iteration (1.7) and equations (2.2-2.3) ,(3.5-3.6), and assumptions, we obtain

**Case 1** For $0 \leq t < 1$

$$\begin{aligned}
&\| z_k(t) - \overline{z}_k(t) \| \\
&= \| (F\omega_k)(t) - (\overline{F}\overline{\omega}_k)(t) \| \\
&\leq \| T(t) \| [\omega_0 - g(\omega_k)] + \int_0^t T(t-s) f(s, \omega_k(s), \phi(s)) ds \\
&\quad - T(t)[\overline{\omega}_0 - g(\overline{\omega}_k)] - \int_0^t T(t-s) f(s, \overline{\omega}_k(s), \phi(s)) ds \\
&\leq \mathcal{M} \| \omega_0 - \overline{\omega}_0 \| + \mathcal{M} \int_0^t \| f(s, \omega_k(s), \phi(s)) - f(s, \overline{\omega}_k(s), \phi(s)) \| ds \\
&\leq \mathcal{M} \| \omega_0 - \overline{\omega}_0 \| | + \mathcal{MG} \| \omega_k - \overline{\omega}_k \| + (\mathcal{ML}b) \| \omega_k - \overline{\omega}_k \| \\
&\leq \mathcal{M} \| \omega_0 - \overline{\omega}_0 \| + (\mathcal{MG} + \mathcal{ML}b) \| \omega_k - \overline{\omega}_k \| \\
&\leq \mathcal{M} \| \omega_0 - \overline{\omega}_0 \| + \Delta_1 \| \omega_k - \overline{\omega}_k \| \\
&\leq \mathcal{M} \| \omega_0 - \overline{\omega}_0 \| + \Delta \| \omega_k - \overline{\omega}_k \| \\
&\leq \mathcal{M}^* + \Delta \| \omega_k - \overline{\omega}_k \|_B \quad (3.8)
\end{aligned}$$

where

$\mathcal{M}^* = \mathcal{M} \| \omega_0 - \overline{\omega}_0 \|$ and $(\mathcal{MG} + \mathcal{ML}b) = \Delta_1$

and

**Case 2** for $1 \leq t < b$,

$\| z_k(t) - \bar{z}_k(t) \|$

$= \| (F\omega_k)(t) - (\bar{F}\bar{\omega}_k)(t) \|$

$\leq \| T(t)[\omega_0 - g(\omega_k)] + \int_0^1 T(t-s) f\big(s, \omega_k(s), \phi(s)\big) ds$

$$+ \int_1^t T(t-s) f(s, \omega_k(s), \omega_k(s-1)) ds$$

$-T(t)[\bar{\omega}_0 - g(\bar{\omega}_k)] - \int_0^1 T(t-s) f\big(s, \bar{\omega}_k(s), \phi(s)\big) ds$

$- \int_1^t T(t-s) f(s, \bar{\omega}_k(s), \bar{\omega}_k(s-1)) ds \|$

$\leq \mathcal{M} \| \omega_0 - \bar{\omega}_0 \| + \mathcal{M}\mathcal{G} \| \omega_k - \bar{\omega}_k \| + (\mathcal{M}\mathcal{L}) \| \omega_k - \bar{\omega}_k \|$

$$+ (2\alpha\mathcal{M}\mathcal{L}b) \| \omega_k - \bar{\omega}_k \|$$

$\leq \mathcal{M} \| \omega_0 - \bar{\omega}_0 \| + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha) \| \omega_k - \bar{\omega}_k \|$

$\leq \mathcal{M} \| \omega_0 - \bar{\omega}_0 \| + \Delta_2 \| \omega_k - \bar{\omega}_k \|$

$\leq \mathcal{M} \| \omega_0 - \bar{\omega}_0 \| + \Delta \| \omega_k - \bar{\omega}_k \|$

where $\Delta = \max\{\Delta_1, \Delta_2\}$

$(\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}b) = \Delta_1$ and $\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha = \Delta_2$ and $\Delta < 1$.

Taking supremum

$$\| z_k - \bar{z}_k \|_B \leq \mathcal{M}^* + \Delta \| \omega_k - \bar{\omega}_k \|_B \qquad (3.9)$$

where

$$\mathcal{M}^* = \mathcal{M} \| \omega_0 - \bar{\omega}_0 \| \qquad (3.10)$$

and

$\| y_k(t) - \bar{y}_k(t) \| = \| (1 - \xi_k)(z_k(t) - \bar{z}_k(t)) + \xi_k((Fz_k)(t) - (\bar{F}\bar{z}_k)(t)) \|$

$$\leq [(1 - \xi_k) \| (z_k(t) - \bar{z}_k(t) \| + \xi_k \| (Fz_k)(t) - (\bar{F}\bar{z}_k)(t) \|] \qquad (3.11)$$

Hence, by taking supremum in the inequality (3.11) and then use the idea from (3,9) to get

$\| y_k - \bar{y}_k \|_B \leq [(1 - \xi_k) \| z_k - \bar{z}_k \|_B + \xi_k \| Fz_k - \bar{F}\bar{z}_k \|_B$

$\leq (1 - \xi_k) \| z_k - \bar{z}_k \|_B + \xi_k[\mathcal{M}^* + \Delta \| z_k - \bar{z}_k \|_B]$

$= \xi_k\mathcal{M}^* + [1 - \xi_k(1 - \Delta)] \| z_k - \bar{z}_k \|_B$

$\leq \xi_k\mathcal{M}^* + [1 - \xi_k(1 - \Delta)][\mathcal{M}^* + \Delta \| \omega_k - \bar{\omega}_k \|_B]$

$$\leq \xi_k\mathcal{M}^* + \mathcal{M}^* + \Delta[1 - \xi_k(1 - \Delta)] \| \omega_k - \bar{\omega}_k \|_B \qquad (3.12)$$

Therefore, using the idea from (??) and (??) along with hypotheses $\bar{\Omega} < 1$, and $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$., the resulting inequality becomes

$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B = \| Fy_k - \bar{F}\bar{y}_k \|_B$

$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq \mathcal{M}^* + \Delta \| y_k - \bar{y}_k \|_B$

$\leq \mathcal{M}^* + \Delta \| y_k - \bar{y}_k \|_B$

$\leq \mathcal{M}^* + \Delta[\xi_k\mathcal{M}^* + \mathcal{M}^* + \Delta[1 - \xi_k(1 - \Delta)] \| \omega_k - \bar{\omega}_k \|_B]$

$\leq 2\mathcal{M}^* + \xi_k\mathcal{M}^* + [1 - \xi_k(1 - \Delta)] \| \omega_k - \bar{\omega}_k \|_B$

$\leq \{2\xi_k\}(2\mathcal{M}^*) + \xi_k\mathcal{M}^* + [1 - \xi_k(1 - \Delta)] \| \omega_k - \bar{\omega}_k \|_B$

$$\leq [1 - \xi_k(1 - \Delta)] \| \omega_k - \bar{\omega}_k \|_B + \xi_k(1 - \Delta)\frac{5\mathcal{M}^*}{(1 - \bar{\Omega})} \qquad (3.13)$$

we denote

$$\beta_k = \| \omega_k - \bar{\omega}_k \|_B \geq 0$$

$\mu_k = \xi_k(1-\Delta) \in (0,1)$

$\gamma_k = \frac{5\mathcal{M}^*}{(1-\Delta)} \geq 0$

The assumption $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$ implies $\{\xi_k\}_{k=0}^{\infty} = \infty$. Now, it can be easily seen that (3.13) satisfies all the conditions of Lemma (1) and hence we have

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k$$

$$0 \leq \lim_{k\to\infty} \parallel \omega_k - \overline{\omega}_k \parallel_B \leq \frac{5\mathcal{M}^*}{(1-\Delta)} \qquad (3.14)$$

using the assumptions $\lim_{k\to\infty}\omega_k = \omega$ ,$\lim_{k\to\infty}\overline{\omega}_k = \overline{\omega}$ , we get from (3.14) that

$$\parallel \omega - \overline{\omega} \parallel_B \leq \frac{5\mathcal{M}^*}{1-\Delta} \qquad (3.15)$$

where

$$\mathcal{M}^* = \mathcal{M} \parallel \omega_0 - \overline{\omega}_0 \parallel_B$$

which shows that the dependency of solutions of IVPs (1.4)-(1.6) and (1.4) with the conditions (3.2) on given initial data.

□

## 4 Continuous dependency on function $f$ and $\overline{f}$

In this section, we shall deal with continuous dependence of solution of the problem (1.4) on the initial functions involved therein

$$\omega'(t) + A\omega(t) = f(t, \omega(t), \omega(t-1)),$$
$$\omega(t-1) = \phi(t) \quad (0 \leq t < 1),$$
$$\omega(0) + g(t) = \omega_0$$

$$\omega'(t) + A\omega(t) = \overline{f}(t, \omega(t), \omega(t-1)), \qquad (4.1)$$
$$\omega(t-1) = \phi(t) \quad (0 \leq t < 1), \qquad (4.2)$$
$$\omega(0) + g(t) = \omega_0 \qquad (4.3)$$

where $\overline{f}$ is defined as f. Suppose $\omega(t)$ and $\overline{\omega}(t)$ are solutions of (1.4-1.6) and (4.1-4.3) respectively.Now ,examining the procedure which occurs in the proof of Theorem 2.1.we establish function $F\omega$ which acts as a operator for (1.4-1.6).also we define $\overline{F\omega}$ which acts as a operator for (4.1-4.3).

**Case 1** for $0 \leq t < 1$

$$F\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^t T(t-s)f\big(s, \omega(s), \phi(s)\big)ds \qquad (4.4)$$

**Case 2** for $1 \leq t < b$,

$$F\omega(t) = T(t)[\omega_0 - g(\omega_k)] + \int_0^1 T(t-s)f\big(s, \omega(s), \phi(s)\big)ds$$
$$+ \int_1^t T(t-s)f(s, \omega(s), \omega(t-1)))ds \qquad (4.5)$$

Also define

**Case 1** for $0 \le t < 1$

$$\overline{F}\overline{\omega}(t) = T(t)[\omega_0 - g(\overline{\omega})] + \int_0^t T(t-s)\overline{f}(s, \overline{\omega}(s), \phi(s))ds) \qquad (4.6)$$

**Case 2** for $1 \le t < b$,

$$\overline{F}\overline{\omega}(t) = T(t)[\omega_0 - g(\overline{\omega})] + \int_0^1 T(t-s)\overline{f}\big(s, \overline{\omega}(s), \phi(s)\big)ds$$
$$+ \int_1^t T(t-s)\overline{f}(s, \overline{\omega}(s), \overline{\omega}(t-1)))ds \qquad (4.7)$$

**Theorem 4.1** *Consider the sequences* $\{\omega_k\}_{k=0}^{\infty}$ *and* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *generated by new three steps iteration method associated with operators F in (2.2-2.3) and* $\overline{F}$ *in (4.6-4.7) respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in [0, 1] satisfying* $\frac{1}{2} \le \xi_k \; \forall k \in \mathbb{N} \cup \{0\}$.

*Assuming that*

*every requirement of main theorem is met with* $\omega(t)$ and $\overline{\omega}(t)$ are solution of (1.4-1.6) and (4.1-4.3) *respectively.*

*There is nonnegative constant* $\epsilon$ *such that*

$$\| f(t, u_1, u_2) - \overline{f}(t, u_1, u_2) \| \le \epsilon, \forall\, t \in J \qquad (4.8)$$

*If the sequence* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *converges to* $\overline{\omega}$, *then we have*

$$\| \omega - \overline{\omega} \|_B \le \frac{5\mathcal{M}\epsilon(1+b)}{(1-\Delta)} \qquad (4.9)$$

*where* $\Delta = max\{\Delta_1, \Delta_2\} = max\,\{(\mathcal{MG} + \mathcal{ML}b), (\mathcal{MG} + \mathcal{ML} + 2\mathcal{ML}b\alpha)\}$ *and* $\Delta < 1$.

**Proof:** Suppose the sequences $\{\omega_k\}_{k=0}^{\infty}$ and $\{\overline{\omega}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators F in (2.2-2.3) and $\overline{F}$ in (4.6-4.7) respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in [0, 1] satisfying $\frac{1}{2} \le \xi_k \; \forall k \in \mathbb{N} \cup \{0\}$.

From iteration (1.7) and equations *(2.2-2.3)* , *(4.6-4.7)* and hypotheses, we obtain

**Case 1** For $0 \le t < 1$

$$\| z_k(t) - \overline{z}_k(t) \|$$

$$= \| (F\omega_k)(t) - (\overline{F}\overline{\omega}_k)(t) \|$$

$$= \| T(t)[\omega_0 - g(\omega_k)] + \int_0^t T(t-s)f\big(s, \omega_k(s), \phi(s)\big)ds - T(t)[\omega_0 - g(\overline{\omega}_k)]$$
$$- \int_0^t T(t-s)\overline{f}(s, \overline{\omega}_k(s), \phi(s))ds) \|$$

$$\le \mathcal{MG} \| \omega_k - \overline{\omega}_k \| + \int_0^t \mathcal{M} \| f(s, \omega_k(s), \phi(s)) - \overline{f}(s, \overline{\omega}_k(s), \phi(s)) \| ds$$

$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \int_0^t \mathcal{M} \parallel f(s, \omega_k(s), \phi(s)) - f(s, \overline{\omega}_k(s), \phi(s))$

$+ f(s, \overline{\omega}_k(s), \phi(s)) - \overline{f}(s, \overline{\omega}_k(s), \phi(s)) \parallel ds$

$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \int_0^t \mathcal{M} \parallel f\big(s, \omega_k(s), \phi(s)\big) - f(s, \overline{\omega}_k(s), \phi(s)) \parallel ds$

$+ \int_0^t \mathcal{M} || f(s, \overline{\omega}_k(s), \phi(s)) - \overline{f}(s, \overline{\omega}_k(s), \phi(s)) \parallel ds$

$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\mathcal{L}\mathrm{b} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\epsilon b$

$\leq \mathcal{M}\epsilon b + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}\mathrm{b}) \parallel \omega_k - \overline{\omega}_k \parallel$

$\leq \mathcal{M}\epsilon b + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}\mathrm{b}) \parallel \omega_k - \overline{\omega}_k \parallel$

$\parallel z_k(t) - \overline{z}_k(t) \parallel \leq \mathcal{M}\epsilon b + \Delta_1 \parallel \omega_k - \overline{\omega}_k \parallel$

where $(\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}\mathrm{b}) = \Delta_1$

By taking supremum

$$\parallel z_k - \overline{z}_k \parallel_B \leq \mathcal{M}\epsilon(1 + b) + \Delta_1 \parallel \omega_k - \overline{\omega}_k \parallel_B \quad . \qquad (4.10)$$

**Case 2** for $1 \leq t < b$,

$\parallel z_k(t) - \overline{z}_k(t) \parallel$

$= \parallel (F\omega_k)(t) - (\overline{F}\overline{\omega}_k)(t) \parallel$

$= \parallel T(t)[\omega_0 - g(\omega_k)] + \int_0^1 T(t-s) f\big(s, \omega_k(s), \phi(s)\big) ds$

$+ \int_1^t T(t-s) f(s, \omega_k(s), \omega_k(s-1)) ds$

$- T(t)[\omega_0 - g(\overline{\omega}_k)] + \int_0^1 T(t-s) \overline{f}\big(s, \overline{\omega}_k(s), \phi(s)\big) ds$

$- \int_1^t T(t-s) \overline{f}(s, \overline{\omega}_k(s), \overline{\omega}_k(s-1)) ds ||$

$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \int_0^1 \mathcal{M} \parallel f(s, \omega_k(s), \phi(s)) - f\big(s, \overline{\omega}_k(s), \phi(s)\big) \parallel ds$

$+ \int_0^1 \mathcal{M} \parallel f\big(s, \overline{\omega}_k(s), \phi(s)\big) - \overline{f}\big(s, \overline{\omega}_k(s), \phi(s)\big) \parallel ds$

$+ \int_1^t \mathcal{M} \parallel f(s, \omega_k(s), \omega_k(s-1)) - f(s, \overline{\omega}_k(s), \overline{\omega}_k(s-1)) \parallel ds$

$+ \int_1^t \mathcal{M} \parallel f(s, \overline{\omega}_k(s), \overline{\omega}_k(s-1)) - \overline{f}(s, \overline{\omega}_k(s), \overline{\omega}_k(s-1)) \parallel ds$

$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\epsilon(1 + b) + (\mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$

$\leq \mathcal{M}\epsilon(1 + b) + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$

$\leq \mathcal{M}\epsilon(1 + b) + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$

$\| z_k(t) - \bar{z}_k(t) \| \leq \mathcal{M}\epsilon(1+b) + \Delta_2 \| \omega_k - \bar{\omega}_k \|$

$\| z_k(t) - \bar{z}_k(t) \| \leq Q + \Delta_2 \| \omega_k - \bar{\omega}_k \|$ (4.11)

where $(\mathcal{MG} + \mathcal{ML} + 2\mathcal{ML}b\alpha) = \Delta_2$.

$\Delta = \max\{\Delta_1, \Delta_2\} = \Delta_2 = (\mathcal{MG} + \mathcal{ML} + 2\mathcal{ML}b\alpha)$ and $Q = \mathcal{M}\epsilon(1+b)$

By taking supremum

$\| z_k - \bar{z}_k \|_B \leq Q + \Delta \| \omega_k - \bar{\omega}_k \|_B$

$\| z_k - \bar{z}_k \|_B \leq Q + \Delta \| \omega_k - \bar{\omega}_k \|_B$

Similarly it is seen that

$\| y_k - \bar{y}_k \| \leq \| (1-\xi_k)z_k + \xi_k F z_k - ((1-\xi_k)\bar{z}_k + \xi_k \overline{F}\bar{z}_k) \|$ (4.12)

$\leq \| (1-\xi_k)(z_k - \bar{z}_k) + \xi_k((Fz_k) - (\overline{F}\bar{z}_k)) \|$

$\leq (1-\xi_k) \| z_k - \bar{z}_k \| + \xi_k \| (Fz_k) - (\overline{F}\bar{z}_k) \|$

$\leq (1-\xi_k) \| z_k - \bar{z}_k \| + \xi_k[Q + \Delta \| z_k - \bar{z}_k \|]$

$\leq (1-\xi_k)(Q + \Delta \| \omega_k - \bar{\omega}_k \|) + \xi_k[Q + \Delta \| z_k - \bar{z}_k \|]$

$\leq (1-\xi_k)(Q + \Delta \| \omega_k - \bar{\omega}_k \|) + \xi_k [Q + \Delta(Q + \Delta \| \omega_k - \bar{\omega}_k \|)]$

$\leq \Delta \| \omega_k - \bar{\omega}_k \| [1 - \xi_k (1-\Delta)] + Q + \xi_k Q$ (4.13)

Now by taking supremum in above inequality,we obtain

$$\| y_k - \bar{y}_k \|_B \leq \Delta \| \omega_k - \bar{\omega}_k \|_B [1 - \xi_k (1-\Delta)] + Q + \xi_k Q \quad (4.14)$$

Therefore, using the idea from (4.10-4.11) and (4.14) along with hypotheses $\Delta < 1$, and $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$ the resulting inequality becomes

$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B = \| F y_k - \overline{F}\bar{y}_k \|_B$$
$$\leq Q + \Delta \| y_k - \bar{y}_k \|_B$$
$$\leq Q + \| y_k - \bar{y}_k \|_B \quad \because \Delta < 1$$

$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq Q + \{\Delta \| \omega_k - \bar{\omega}_k \|_B [1 - \xi_k (1-\Delta)] + Q + \xi_k Q\}$$
$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq 2Q + \{\| \omega_k - \bar{\omega}_k \|_B [1 - \xi_k (1-\Delta)] + \xi_k Q\}$$

Since $\Delta < 1$ and $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$

$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq (2\xi_k)[2Q] + \xi_k[Q] + [1 - \xi_k (1-\Delta)] \| \omega_k - \bar{\omega}_k \|_B$$
$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq 5\xi_k Q + [1 - \xi_k (1-\Delta)] \| \omega_k - \bar{\omega}_k \|_B$$

$$\| \omega_{k+1} - \bar{\omega}_{k+1} \|_B \leq [1 - \xi_k (1-\Delta)] \| \omega_k - \bar{\omega}_k \|_B + \xi_k(1-\Delta)\frac{5Q}{(1-\Delta)} \quad (4.15)$$

we denote

$$\beta_k = \| \omega_k - \overline{\omega}_k \|_B \geq 0$$
$$\mu_k = \xi_k(1-\Delta)$$
$$\gamma_k = \frac{5Q}{(1-\Delta)} \geq 0$$

The assumption $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$ implies $\{\xi_k\}_{k=0}^{\infty} = \infty$. Now, it can be easily seen that (4.15) satisfies all the conditions of Lemma (1) and hence we have

$$0 \leq \lim_{k\to\infty} \sup \beta_k \leq \lim_{k\to\infty} \sup \gamma_k (4.16)$$
$$0 \leq \lim_{k\to\infty} \sup \| \omega_k - \overline{\omega}_k \|_B \leq \sup_{k\to\infty} \frac{5Q}{(1-\Delta)}$$
$$0 \leq \lim_{k\to\infty} \sup \| \omega_k - \overline{\omega}_k \|_B \leq \frac{5Q}{(1-\Delta)}.$$

Using the assumptions $\lim_{k\to\infty}\omega_k = \omega$ ,$\lim_{k\to\infty}\overline{\omega}_k = \overline{\omega}$ , we get from (4.16) that

$$\| \omega - \overline{\omega} \|_B \leq \frac{5Q}{(1-\Delta)}$$

where $Q = \mathcal{M}\epsilon(1+b)$ and $\Delta = (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L} + 2\mathcal{M}\mathcal{L}b\alpha)$
Using the assumptions $\lim_{k\to\infty}\omega_k = \omega$ ,$\lim_{k\to\infty}\overline{\omega}_k = \overline{\omega}$ , we get

$$\| \omega - \overline{\omega} \|_B \leq \frac{5\mathcal{M}\epsilon(1+b)}{(1-\Delta)}$$

which shows that the dependency of solutions of IVP (1.4)-(1.6) on both the function involved from the right hand side of the given equation and initial data.

□

**Remark:** The inequality (43) relates the solutions of the problems (1.4)-(1.6) and (4.1)-(4.3) in the sense that, if $f$ and $\overline{f}$are close as $\epsilon \to 0$ , then not only the solutions of the problems (1.4)-(1.6) and (4.1)-(4.3) are close to each other (i.e. $\| \omega - \overline{\omega} \|_B \to 0$), but also depends continuously on the functions involved therein and initial data.

## 5 Continuous dependency on Parameter via new three steps iteration

We next consider the following problems Let $\omega(t), \overline{\omega}(t)$

$$\omega'(t) + A\omega(t) = f(t, \omega(t), \omega(t-1), \theta_1), \qquad (5.1)$$
$$\omega(t-1) = \phi(t) \quad (0 \leq t < 1), \qquad (5.2)$$
$$\omega(0) + g(t) = \omega_0 \qquad (5.3)$$

and

$\omega'(t) + A\omega(t) = \overline{f}(t, \omega(t), \omega(t-1), \theta_2),$ (5.4)

$\omega(t-1) = \phi(t) \quad (0 \le t < 1),$ (5.5)

$\omega(0) + g(t) = \omega_0$ (5.6)

Where constants $\theta_1, \theta_2$ are any real number which considered as a parameters .Let $\omega(t), \overline{\omega}(t)$and completing the steps from the previous theorem proof of, define the new function as a operators for the equations (5.1) and (5.4) respectively, We Define

**Case 1** for $0 \le t < 1$

$F\omega(t) = T(t)[\omega_0 - g(\omega)] + \int_0^t T(t-s)f(s, \omega(s), \phi(s), \theta_1)ds$ (5.7)

**Case 2** for $1 \le t < b$,

$F\omega(t) = T(t)[\omega_0 - g(\omega_k)] + \int_0^1 T(t-s)f(s, \omega(s), \phi(s), \theta_1)ds$
$+ \int_1^t T(t-s)f(s, \omega(s), \omega(t-1), \theta_1)ds$ (5.8)

also define

**Case 1** for $0 \le t < 1$

$\overline{F}\overline{\omega}(t) = T(t)[\omega_0 - g(\overline{\omega})] + \int_0^t T(t-s)\overline{f}(s, \overline{\omega}(s), \phi(s), \theta_2)ds)$ (5.9)

**Case 2** for $1 \le t < b$,

$\overline{F}\overline{\omega}(t) = T(t)[\omega_0 - g(\overline{\omega})] + \int_0^1 T(t-s)\overline{f}(s, \overline{\omega}(s), \phi(s), \theta_2)ds$
$+ \int_1^t T(t-s)\overline{f}(s, \overline{\omega}(s), \overline{\omega}(t-1), \theta_2)ds$ (5.10)

The upcoming theorem demonstrates the continuous dependency (CD) of solutions on parameters.

**Theorem 5.1** *Suppose the sequences* $\{\omega_k\}_{k=0}^{\infty}$ *and* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *generated by new three steps iteration method associated with operators* $F$ *in* $(5.1)-(5.3)$ *and and* $\overline{F}$ *in* $(5.4)-(5.6)$*,respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in [0, 1] satisfying* $\frac{1}{2} \le \xi_k\ \forall k \in \mathbb{N} \cup \{0\}$.

*Suppose that*

$\omega(t)$*) and* $\overline{\omega}(t)$*are solution of (5.1)-(5.3) and (5.4)-(5.6) respectively.*

*there exist constants* $\mathcal{L}_1 > 0$ *such that the function* $\overline{f}$ *satisfy the conditions*

$\| \overline{f}(s, x, y, \theta_1) - \overline{f}(s, \overline{x}, \overline{y}, \theta_2) \| \le \mathcal{L}_1(\| x - \overline{x} \| + \| y - \overline{y} \| + |\theta_1 - \theta_2|$ *(5.11)*

*Then If the sequence* $\{\overline{\omega}_k\}_{k=0}^{\infty}$ *converges to* $\overline{\omega}$*, then we followig this inequality*

$\| \omega - \overline{\omega} \|_B \le \frac{5\mathcal{M}\mathcal{L}_1[1+b]|\theta_1-\theta_2|}{(1-\bar{\Delta})}$ *(5.12)*

*where* $\bar{\Delta} = max\,\{\bar{\Delta}_1, \bar{\Delta}_2\} = max\,\{\,\mathcal{MG} + \mathcal{M}b\mathcal{L}_1, \mathcal{MG} + \mathcal{ML}_1 + 2\mathcal{M}b\mathcal{L}_1\alpha\}$
*and* $\bar{\Delta} < 1.$

**Proof:-** Suppose the sequences $\{\omega_k\}_{k=0}^{\infty}$ and $\{\overline{\omega}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators F in (5.1) − (5.3) and and $\overline{F}$ in (5.4) − (5.6),respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in [0, 1] satisfying $\frac{1}{2} \le \xi_k\ \forall k \in \mathbb{N} \cup \{0\}$.

**Case 1** For $0 \le t < 1$

Consider
$\| z_k(t) - \bar{z}_k(t) \|$

$$=\| (F\omega_k)(t) - (\overline{F\omega}_k)(t) \| \qquad (5.13\,)$$

$$=\| T(t)[\omega_0 - g(\omega_k)] + \int_0^t T(t-s)f(s,\omega_k(s),\phi(s),\theta_1)ds - T(t)[\omega_0 - g(\overline{\omega}_k)]$$
$$- \int_0^t T(t-s)\overline{f}(s,\overline{\omega}_k(s),\phi(s),\theta_2)ds) \|$$

$$\le \mathcal{MG} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\int_0^t \parallel \overline{f}(s,\omega_k(s),\phi(s),\theta_1) - \overline{f}(s,\overline{\omega}_k(s),\phi(s),\theta_2) \parallel ds$$
$$\le \mathcal{MG} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\mathrm{b}\mathcal{L}_1(\parallel \omega_k - \overline{\omega}_k \parallel + |\theta_1 - \theta_2|)$$
$$\le \mathcal{M}\mathrm{b}\mathcal{L}_2|\theta_1 - \theta_2| + (\mathcal{MG} + \mathcal{M}\mathrm{b}\mathcal{L}_1) \parallel \omega_k - \overline{\omega}_k \parallel$$

$$\le W_1 + (\mathcal{MG} + \mathcal{M}\mathrm{b}\mathcal{L}_1) \parallel \omega_k - \overline{\omega}_k \parallel$$
We get ,$\| z_k(t) - \bar{z}_k(t) \| \le W_1 + \bar{\Delta}_1 \parallel \omega_k - \overline{\omega}_k \parallel$ (5.14)

Where $(\mathcal{MG} + \mathcal{M}\mathrm{b}\mathcal{L}_1) = \bar{\Delta}_1$ and $W_1 = \mathcal{M}\mathrm{b}\mathcal{L}_1|\theta_1 - \theta_2|$ (5.15)
By taking supremum

$$\| z_k(t) - \bar{z}_k(t) \|_B \le W_1 + \bar{\Delta}_1 \parallel \omega_k - \overline{\omega}_k \parallel_B \qquad (5.16)$$

**Case 2** for $1 \le t < b$,
$\| z_k(t) - \bar{z}_k(t) \|$

$$=\| (F\omega_k)(t) - (\overline{F\omega}_k)(t) \|$$

$$=\| T(t)[\omega_0 - g(\omega_k)] + \int_0^1 T(t-s)\overline{f}(s,\omega_k(s),\phi(s),\theta_1)ds$$

$$+\int_1^t T(t-s)\overline{f}(s,\omega_k(s),\omega_k(t-1),\theta_1)ds$$
$$-T(t)[\omega_0 - g(\overline{\omega_k})] - \int_0^1 T(t-s)\overline{f}(s,\overline{\omega_k}(s),\phi(s),\theta_2)ds$$
$$-\int_1^t T(t-s)\overline{f}(s,\overline{\omega_k}(s),\overline{\omega_k}(t-1),\theta_2)ds \parallel$$
$$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\int_0^1 \parallel f(s,\omega_k(s),\phi(s),\theta_1) - \overline{f}(s,\overline{\omega}_k(s),\phi(s),\theta_2) \parallel ds$$
$$+\mathcal{M}\int_1^t \parallel f(s,\omega_k(s),\phi(s),\theta_1) - \overline{f}(s,\overline{\omega}_k(s),\phi(s),\theta_2) \parallel ds$$
$$\leq \mathcal{M}\mathcal{G} \parallel \omega_k - \overline{\omega}_k \parallel + \mathcal{M}\mathcal{L}_1(\parallel \omega_k - \overline{\omega}_k \parallel + |\theta_1 - \theta_2|)$$
$$+\mathcal{M}\mathrm{b}\mathcal{L}_1(\parallel \omega_k - \overline{\omega}_k \parallel + \parallel \omega_k(t-1) - \overline{\omega_k}(t-1) \parallel + |\theta_1 - \theta_2|)$$
$$\leq \{\mathcal{M}\mathcal{L}_2|\theta_1 - \theta_2| + \mathcal{M}\mathrm{b}\mathcal{L}_1|\theta_1 - \theta_2|\} + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$$
$$\leq \{(\mathcal{M}\mathcal{L}_1 + \mathcal{M}\mathrm{b}\mathcal{L}_1)|\theta_1 - \theta_2|\} + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$$
$$\leq \{(\mathcal{M}\mathcal{L}_1 + \mathcal{M}\mathrm{b}\mathcal{L}_1)|\theta_1 - \theta_2|\} + (\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha) \parallel \omega_k - \overline{\omega}_k \parallel$$
$$\leq W_2 + \overline{\Delta}_2 \parallel \omega_k - \overline{\omega}_k \parallel$$
$$\parallel z_k(t) - \overline{z}_k(t) \parallel \leq W_2 + \overline{\Delta}_2 \parallel \omega_k - \overline{\omega}_k \parallel \qquad (5.17)$$

Where $(\mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha) = \overline{\Delta}_2$ and $W_2 = \mathcal{M}\mathcal{L}_1[1+\mathrm{b}]|\theta_1 - \theta_2|$ (5.18)
By taking supremum

$$\parallel z_k(t) - \overline{z}_k(t) \parallel_B \leq W_2 + \Delta_2 \parallel \omega_k - \overline{\omega}_k \parallel_B$$
$$\overline{\Delta} = \max\{\Delta_1, \Delta_2\} = \mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha$$
$$\parallel z_k(t) - \overline{z}_k(t) \parallel_B \leq W + \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel_B \qquad (5.19)$$

Similarly

$$\parallel y_k - \overline{y}_k \parallel \leq \parallel (1-\xi_k)z_k + \xi_k F z_k - \left((1-\xi_k)\overline{z}_k + \xi_k \overline{F}\overline{z}_k\right) \parallel \qquad (5.20)$$
$$\leq \parallel (1-\xi_k)(z_k - \overline{z}_k) + \xi_k((Fz_k) - (\overline{F}\overline{z}_k)) \parallel$$
$$\leq (1-\xi_k) \parallel z_k - \overline{z}_k \parallel + \xi_k \parallel (Fz_k) - (\overline{F}\overline{z}_k) \parallel$$
$$\leq (1-\xi_k) \parallel z_k - \overline{z}_k \parallel + \xi_k[\mathrm{W} + \overline{\Delta} \parallel z_k - \overline{z}_k \parallel]$$
$$\leq (1-\xi_k)(\mathrm{W} + \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel) + \xi_k[\mathrm{W} + \overline{\Delta} \parallel z_k - \overline{z}_k \parallel]$$
$$\leq (1-\xi_k)(\mathrm{W} + \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel) + \xi_k\,[W + \overline{\Delta}(\mathrm{W} + \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel)]$$
$$\leq \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel [1 - \xi_k\,(1-\overline{\Delta})] + \mathrm{W} + \xi_k\,W$$

Now by taking supremum in above inequality,we obtain

$$\parallel y_k - \overline{y}_k \parallel_B \leq \overline{\Delta} \parallel \omega_k - \overline{\omega}_k \parallel_B [1 - \xi_k\,(1-\overline{\Delta})] + \mathrm{W} + \xi_k\,W \qquad (5.21)$$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B = \| Fy_k - \overline{F}\overline{y}_k \|_B \quad (5.22)$$

$$\leq W + \overline{\Delta} \| y_k - \overline{y}_k \|_B \quad (5.23)$$

$$\leq W + \| y_k - \overline{y}_k \|_B \quad \because \overline{\Delta} < 1$$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B \leq W + \{\overline{\Delta} \| \omega_k - \overline{\omega}_k \|_B [1 - \xi_k (1 - \overline{\Delta})] + W + \xi_k W\} \quad (5.24)$$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B \leq 2W + \{\| \omega_k - \overline{\omega}_k \|_B [1 - \xi_k (1 - \overline{\Delta})] + \xi_k W\}$$

Since $\overline{\Delta} < 1$ and $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B \leq (2\xi_k)[2W] + \xi_k[W] + [1 - \xi_k (1 - \overline{\Delta})] \| \omega_k - \overline{\omega}_k \|_B$$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B \leq 5\xi_k W + [1 - \xi_k (1 - \overline{\Delta})] \| \omega_k - \overline{\omega}_k \|_B$$

$$\| \omega_{k+1} - \overline{\omega}_{k+1} \|_B \leq [1 - \xi_k (1 - \overline{\Delta})] \| \omega_k - \overline{\omega}_k \|_B + \xi_k (1 - \overline{\Delta}) \frac{5W}{(1-\overline{\Delta})} \quad (5.25)$$

we denote

$$\beta_k = \| \omega_k - \overline{\omega}_k \|_B \geq 0$$

$$\mu_k = \xi_k (1 - \overline{\Delta})$$

$$\gamma_k = \frac{5Q}{(1-\overline{\Delta})} \geq 0$$

The assumption $\frac{1}{2} \leq \xi_k \ \forall k \in \mathbb{N} \cup \{0\}$ implies $\{\xi_k\}_{k=0}^{\infty} = \infty$. Now, it can be easily seen that (5.25) satisfies all the conditions of Lemma (1) and hence we have

$$0 \leq \lim_{k\to\infty} \sup \beta_k \leq \lim_{k\to\infty} \sup \gamma_k$$

$$0 \leq \lim_{k\to\infty} \sup \| \omega_k - \overline{\omega}_k \|_B \leq \sup_{k\to\infty} \frac{5W}{(1-\overline{\Delta})}$$

$$0 \leq \lim_{k\to\infty} \sup \| \omega_k - \overline{\omega}_k \|_B \leq \frac{5W}{(1-\overline{\Delta})} \quad (5.26)$$

using the assumptions $\lim_{k\to\infty} \omega_k = \omega$ ,$\lim_{k\to\infty} \overline{\omega}_k = \overline{\omega}$ , we get that

$$\| \omega - \overline{\omega} \|_B \leq \frac{5W}{(1-\overline{\Delta})} \quad (5.27)$$

where $W = \mathcal{M}\mathcal{L}_1[1 + \mathrm{b}]|\theta_1 - \theta_2|$ and $\overline{\Delta} = \mathcal{M}\mathcal{G} + \mathcal{M}\mathcal{L}_1 + 2\mathcal{M}\mathrm{b}\mathcal{L}_1\alpha$

Using the assumptions $\lim_{k\to\infty} \omega_k = \omega$ ,$\lim_{k\to\infty} \overline{\omega}_k = \overline{\omega}$ , we get

$$\| \omega - \overline{\omega} \|_B \leq \frac{5\mathcal{M}\mathcal{L}_1[1+\mathrm{b}]|\theta_1-\theta_2|}{(1-\overline{\Delta})} \quad (5.28)$$

which shows the dependence of solutions of the problem (1.4)-(1.6) on parameters $\theta_1, \theta_2$.

**Remark :** The final outcome pertaining to the property of a solution called "dependence of solutions on parameters ". The parameters in this case are scalars.Observe that there are no parameters in the initial conditions. A key element of numerous problems in physics is the reliance on parameters.

## 2. CONCLUSIONS

In the beginning, we demonstrated the main result, which focuses on the existence and uniqueness of the solution of an abstract nonlinear delay differential equation with nonlocal condition by new three steps iteration method. Next,We addressed about several aspects of properties and behaviour of solutions like continuous dependency(CD) on the initial data, functions involve therein. closeness of solutions, and dependence on parameters.

**Acknowledgement:** The authors gratefully acknowledge the valuable support and funding provided through the VCRMS scheme run by Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon

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